Aziz Azizov

SINGLE CHANNEL INVARIANCE PRINCIPLE in DYNAMICS OF MEASURING SYSTEMS

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The physical single-channel principle of invariance based on the statement and solution of the extended problem of dynamic measurements is considered. In this formulation of the problem, both the signal to be measured and the parameters characterizing dynamic properties of the linear measuring system are considered unknown.

This principle of invariance allows us to solve a number of complex tasks of dynamic measurements, including the most important problem of elimination of the influence of parametric phenomena on measurement accuracy.

The main applied focus of the monograph is the field of measurement of non-electrical parameters of technological processes. The above mentioned methods and results can also be used to solve the general problem of determining the reaction of the input action object for linear non-stationary dynamic objects of any nature.

For scientists and technical engineers.

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INTRODUCTION

When setting an extended dynamic measurement task, both the signal being measured and the parameters of the linear non-stationary measuring system of the given structure are considered unknown. This essentially means combining the actual measurement problem and the problem of parametric identification of the measuring system.

Approximation in the extended problem of the measured signal and unknown parameters of the measuring system by polynomials allows us to replace the problem with unknown functions with a problem with unknown constant values. The existence of various methods of reducing the last problem to the solution of some equivalent systems of linear algebraic equations (SLAE) provides the implementation of physical single-channel invariance principle.

The essence of this principle of invariance is that using only the readings of a non-stationary measuring system and the given mathematical model of this system allows us to find independently all unknown constant values (coefficients of approximation), and consequently, all unknown functions - the measured signal and time-variable parameters of the measuring system.

This paper describes in detail the content of the physical single-channel principle of invariance with respect to linear non-stationary measuring systems of a given structure, and considers both measuring systems with lumped parameters, and measuring systems with distributed parameters.

Single-channel invariance principle applied to linear non-stationary measuring systems with lumped unstructured (unknown) parameters is considered separately.

Since the single-channel principle of invariance is implemented in solving the inverse problem, which is an extended measurement problem, the issues of increasing the stability of specific invariance algorithms are considered.

The paper concludes with the consideration of the application of the single-channel principle of invariance in statistical dynamics of measuring systems.

The results of a computer simulation of the measured signals recovery process using single-channel invariance principle are covered. Within the framework of the selected indirect and direct criteria to evaluate the accuracy of the measured signal recovery, the simulation’s results confirm the validity of the main theoretical conclusion: single channel invariance principle allows us to obtain information about measured signals free of parametric distortions. This fact is very important, as the exclusion of the influence of parametric effects significantly increases the accuracy of dynamic measurements.

CHAPTER 1

MATHEMATICAL MODELS OF TYPICAL SUBSYSTEMS THAT ARE SOURCES OF PARAMETRIC PHENOMENA

Parametric phenomena in the field of measurement can be observed both in the subsystems of obtaining primary measuring information - in the so-called measuring converters (MC) of primary measuring information (“sensors”), and in measurement information transformation subsystems - in technical means of functional transformation of measuring information (amplifiers, delay units, compensating elements, integrating and differentiating circuits, auto compensators, etc.).

The main focus of this work is the area of measurement of non-electric quantities, where, as we know, parametric effects are very strong, and in subsystems of obtaining primary measuring information, these effects are disproportionately more significant than in measurement information transformation subsystems. In this regard, below are examples of mathematical models of only typical subsystems for obtaining primary measurement information in the field of measurement of non-electrical quantities.

1.1. SUBSYSTEMS WITH LUMPED PARAMETERS

The following models are limited to the measurement of speed, temperature and pressure of fluids and gases. These models shall use the symbols adopted in the respective areas of measurement.

Flow rate measuring converters (MC)

There are many types of MC flow rate, all based on completely different physical principles of action. Let's delve into the simplest of them, namely, the MC called the “bowl” or “windmill anemometer”. If the friction in parts of the mechanism is neglected, at sufficiently high flow rates, the equation characterizing the dynamic properties of the anemometer as a linear link can be represented as

$$I\,\frac{dw(t)}{dt}+r(t)w(t)=c_0\Omega^2(t),\;\; w(0)=w_0, \tag{1}$$

where $w(t)$ - rotor rotation speed, which is used to evaluate the flow rate; $\Omega(t)$ - flow rate; I - moment of inertia of the anemometer rotor; c_0 - constant, depending on the technological parameters of the anemometer; $r(t)$ - time-variable parameter, depending on viscous friction and measurement conditions, which is the source of parametric phenomena.

Flow temperature measuring converters (MC)

Let's do this with contact methods for measuring the temperature of fluids and gases. In this case, thermocouples, thermistors and various types of resistance thermometers are typical temperature MC (heat sensor).

Let's start with the simplest situation. Suppose that the material of the heat sensor is homogeneous, and during the measurement process there are no temperature gradients inside the heat sensor. Then the equation describing the dynamic properties of these types of heat sensors as linear links will be the equation

$$\frac{du(t)}{dt}+\frac{\alpha_k(t)\cdot S}{c\gamma V}u(t)=\frac{\alpha_k(t)\cdot S}{c\gamma V}\theta(t),\;\; u(0)=u_0, \tag{2}$$

where $u(t)$ - heat sensor temperature; $\theta(t)$ - measured flow temperature; S, V - surface area and volume of heat sensor; c, γ - specific heat capacity and density of heat sensor material; $\alpha_k(t)$ - coefficient of convective heat exchange between the heat sensor and the medium in which the heat sensor is located.

The time variability of the convective heat transfer coefficient is a source of parametric phenomena.

A more complex flow temperature MC is the industrial type of MC. They consist of the above mentioned heat sensors, playing the role of sensitive elements, which are placed in a protective shell. Assuming that the material of the shell of such a MC is homogeneous in its physical properties and there are no temperature gradients in it as well as inside the sensitive element, describing the dynamic properties of this group of heat sensors has the form

$$\frac{d^2u(t)}{dt^2} + [\beta_1 + \beta_2 + \beta_3 \cdot \alpha_k(t)] \frac{du(t)}{dt} + \beta_1 \cdot \beta_3 \cdot \alpha_k(t) \cdot u(t) = \beta_1 \cdot \beta_3 \cdot \alpha_k(t) \cdot \theta(t), \tag{3}$$

$$u(0) = u_0, \quad u'(0) = u_1,$$

where $\beta_1 = \frac{k_0 S_э}{c_э}$; $\beta_2 = \frac{k_0 S_э}{c_{об}}$; $\beta_3 = \frac{S_{об}}{c_{об}}$; $c_э$, $c_{об}$ - the total heat capacity of the heat-sensitive element and shell respectively; k_0 - heat transfer coefficient between the shell and the sensitive element; $S_э$, $S_{об}$ - surface area sensitive element and shell respectively. The notation $u(t)$, $\theta(t)$, $\alpha_k(t)$ has the same meaning as in equation (2).

Flow pressure measuring conductors (MC)

Typical gas flow pressure converters are pressure gauges with an elastic sensitive element. If the inertia of the elastic sensitive element is neglected, the dynamic properties of these MC's will be completely determined by the properties of aerodynamic transmission routes. In this case, the equation describing the dynamic properties of the MC is

$$\tau(p,t)\frac{dp(t)}{dt} + p(t) = P(t), \quad p(0) = p_0, \tag{4}$$

where $\tau(p,t) = \frac{128\mu(t) \cdot v_0 \cdot l_0}{\pi d^4 \cdot p(t)}$; $p(t)$ - the current value of the gas pressure in the monometer cavity, which, according to the above assumption, is reproduced without elastic distortion by an elastic sensitive element, that is, $p(t)$ - the MC readings; $P(t)$ - the measured pressure of the gas stream; v_0 - the volume of gas enclosed in the cavity of the monometer; l_0 and d are the length and diameter of the monometer tube; $\mu(t)$ - the coefficient of dynamic viscosity of the gas, which depends on time and is a source of parametric phenomena.

Note that the model (4) is essentially nonlinear, since the parameter $\tau(p, t)$ contains the function $p(t)$. However, in practice, when evaluating the parameter $\tau(p, t)$, instead of the function $p(t)$, included in its structure, a value from the assumed interval of change of the measured signal $P(t)$ is used.

There may be another consideration of these pressure MC. Suppose that in monometers with an elastic sensitive element, the dynamic properties of the MC are determined only by the inertia of the elastic sensitive element, and the presence of aerodynamic paths can be neglected. Assuming that the elastic element is a membrane-type element, the dynamic properties of MC as a linear element will be described by the equation

$$m\frac{d^2W(t)}{dt^2} + k(t)\frac{dW(t)}{dt} + \bar{c}W(t) = P(t), \quad W(0) = W_0, \quad W'(0) = W_1, \tag{5}$$

where $W(t)$ - current deflection value of the membrane; $P(t)$ - measured pressure; m, $\bar{c}$ - mass and stiffness of the membrane; $k(t)$ - parameter characterizing the damping of the membrane, which is the source of parametric phenomena.

If both the inertia of the aerodynamic trace and the inertia of the elastic sensitive element were to be taken into account when analyzing the dynamic properties of the pressure MC, then it would be necessary to consider equation (4) together with equation (5), replacing function $P(t)$ by function $p(t)$ in the latter.

On the basis of a specific type (1) - (5) of the given models, as a general model of MC with lumped parameters described by ordinary differential equations, in the simplest case it is possible to take equation

$$\frac{d^nY(t)}{dt^n} + \sum_{i=0}^{n-1} a_i(t)\frac{d^iY(t)}{dt^i} = X(t), \quad Y^{(i)}(0) = Y_0^{(i)}, \quad i = 0, 1, \ldots, (n-1), \tag{I}$$

or the equation

$$\frac{d^nY(t)}{dt^n} + \sum_{i=0}^{n-1} a_i(t)\frac{d^iY(t)}{dt^i} = a_0(t)X(t), \quad Y^{(i)}(0) = Y_0^{(i)}, \quad i = 0, 1, \ldots, (n-1), \tag{II}$$

which is determined by a specific area of measurement. In both cases, $Y(t)$ - the output signal, i.e. the readings of MC; $X(t)$ – the measured signal; $a_i(t)$, $i = 0, \ldots, (n-1)$ – time-variable parameters of MC.

In the future we will consider these models separately, as the physical properties of the converters described are different, and when referring to them we call them the first and second model measuring systems with lumped parameters.

1.2. SUBSYSTEMS WITH DISTRIBUTED PARAMETERS

Consideration of MC's with distributed parameters is due to the need for more accurate physical and mathematical descriptions of these objects. Here, we restrict ourselves to providing only models of primary measuring converters of temperature and pressure, since these models quite fully contain the main problems of analysis that arise when studying the dynamic properties of MC's with distributed parameters.

Flow temperature measuring converters

The simplest MC's, from the point of view of the analysis of parametric phenomena, are measuring temperature converters with distributed parameters, so-called rod-type heat sensors. The dynamic properties of these MC's in the linear approximation are described by the boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = a_0^2 \frac{\partial^2 u(x,t)}{\partial x^2} + m(t)[\theta(t) - u(x,t)], \quad 0 < x < l, \quad t > 0 \tag{6}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad u(l,t) = u_{\text{ст}} = \text{const}, \quad u(x,0) = u_0 = \text{const}, \quad m(t) = \frac{\beta}{c\gamma}\alpha_k(t), \tag{7}$$

where $u(x, t)$ - the temperature of the rod at point x at time t; $\theta(t$ - the measured flow temperature; $\alpha_k(t)$ - the coefficient of convective heat transfer between the heat sensor and the medium in which the heat sensor is located; a_0^2, c, γ- coefficient of temperature conductivity, specific heat capacity and density of heat sensor material respectively; β- determining the size of the rod; l - length of the rod; $u_{\text{ст}}$ - the temperature of the wall in which the thermal sensor is fixed.

Depending on the design, as the output value of these thermal sensors, is as a rule, either the temperature of the beginning of the rod, i.e., the temperature at the point $x = 0$, or the average temperature of the rod along the length l_0, determined by the formula

$$u_{\text{ñð}}(t) = \frac{1}{l_0}\int_0^{l_0} u(x,t)dx, \quad l_0 \le l.$$

Obviously, the source of parametric phenomena for these MC's is the convective heat transfer coefficient $\alpha_k(t)$. In the further analyses, model (6) - (7) will be considered the first model of the measuring system with distributed parameters.

As the next group of heat sensors described by the boundary value problem, we will consider measuring temperature converters, the dynamic properties of which in the linear approximation are described by the boundary value task

$$\frac{\partial u(x,y,z,t)}{\partial t} = a_0^2\left[\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right],$$

$$(x,y,z) \in \Omega_0, \quad t > 0; \tag{8}$$

$$\left.\frac{\partial u(x,y,z,t)}{\partial n}\right|_s + H(t)\left[u(x,y,z,t)|_s - \theta(t)\right] = 0, \quad u(x,y,z,0) = u_0 = \text{const}, \tag{9}$$

where $H(t) = \frac{\alpha_k(t)}{\lambda}$; $u(x, y, z, t)$ - the temperature of the bulb fluid at the spatial point (x, y, z) at time t; $\theta(t)$ - the measured flow temperature; λ and a_0^2 are the coefficients of thermal conductivity and thermal diffusivity of the material of the thermal sensor, respectively; $\frac{\partial u(x,y,z,t)}{\partial n}$- derivative of the temperature in the direction of the normal n to the isothermal surface.

Model (8) - (9) describes temperature MC of arbitrary geometrical shape, but made of homogeneous materials in physical properties. Technical execution of temperature MC is usually limited to bodies of canonical forms: unlimited plate (the length and width of the plate is many times greater than its thickness), a ball and an unlimited cylinder (the length of the cylinder is many times greater than its diameter). In this case, the general model (8) - (9) for the specified bodies of canonical forms is converted accordingly. For MC's with the shape of an unlimited plate, model (8) - (9) takes the form of

$$\frac{\partial u(x,t)}{\partial t} = a_0^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < R, \quad t > 0, \tag{10}$$

$$\frac{\partial u(R,t)}{\partial x} + H(t)\left[u(R,t) - \theta(t)\right] = 0, \frac{\partial u(0,t)}{\partial x} = 0, \quad u(x,0) = u_0, \tag{11}$$

where R - half the thickness of the plate; x - the current distance from the arbitrary point of the plate to the plane of symmetry of the plate, which passes through the point $x = 0$ parallel to the plane of the plate surface.

For MC's with the shape of a ball, model (8) - (9) takes the form of:

$$\frac{\partial u(r,t)}{\partial t}=a_0^2\left[\frac{\partial^2 u(r,t)}{\partial r^2}+\frac{2}{r}\frac{\partial u(r,t)}{\partial r}\right],\quad 0<r<R,\quad t>0, \tag{12}$$

$$\frac{\partial u(R,t)}{\partial r}+H(t)\left[u(R,t)-\theta(t)\right]=0,\frac{\partial u(0,t)}{\partial r}=0,\; u(r,0)=u_0, \tag{13}$$

where R is the radius of the ball; r is the current distance from an arbitrary point of the ball to its center.

For MC's with the shape of an unlimited cylinder, model (8) - (9) takes the form of:

$$\frac{\partial u(r,t)}{\partial t}=a_0^2\left[\frac{\partial^2 u(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,t)}{\partial r}\right],\quad 0<r<R,\quad t>0, \tag{14}$$

$$\frac{\partial u(R,t)}{\partial r}+H(t)\left[u(R,t)-\theta(t)\right]=0,\frac{\partial u(0,t)}{\partial r}=0,\; u(r,0)=u_0, \tag{15}$$

where R - the radius of the cylinder; r - the current distance from an arbitrary point of the cylinder to the axis of symmetry of the cylinder passing through the point $r = 0$.

The readings of heat sensors described by these boundary problems correspond to their average temperature $u_v(t)$:

for heat sensors with the shape of an unlimited plate -

$$u_v(t)=\frac{1}{R}\int_0^R u(x,t)dx;$$

for heat sensors with the shape of a ball, -

$$u_v(t)=\frac{3}{R^3}\int_0^R r^2 u(r,t)dr;$$

for thermal receivers with the shape of an unlimited cylinder -

$$u_v(t)=\frac{2}{R^2}\int_0^R ru(r,t)dr.$$

Models of temperature MC's with distributed parameters (8) - (15) are physically and mathematically more accurate descriptions of temperature MC's with concentrated parameters, the dynamic properties of which are described in a rather rough approximation by a simple model (2). In further analysis, model (8) - (9) and its specific cases (10) - (15) will be considered the second model of measuring systems with distributed parameters.

In conclusion of the consideration of this group of MC temperatures with distributed parameters, we should note that there are technical designs of thermal sensors for which the above-mentioned thermometric bodies of canonical forms are not sensitive elements that directly record the temperature of the medium in which they are located, but which serve as so-called substrates. In these technical versions of heat sensors, the sensitive elements that directly record the temperature of the medium are the thinnest metal films, which using special technology, are applied on the surface of non-conductive substrates. The extreme thinness of the films allows us to assume that there are no temperature gradients inside the films, and the temperature of the films coincides with the temperature of the surface of the substrate. Therefore, under these conditions it is possible to ignore the fact of presence of sensitive films on the surface of substrates of heat sensors, and to consider that the readings of heat sensors correspond to the temperature of the substrate surface. Heat sensors technically executed in the described manner are commonly referred to as film heat sensors. In practice, non-conductive substrates of film heat sensors are often performed, also in the form of so-called semi-bounded bodies (semi-bounded cylinders), on the end surface of which the sensitive film is applied.

Flow pressure measuring converters

Let us again turn to the pressure MC - to the monometers with an elastic sensitive element, and assume that the dynamic properties of these MC's are determined only by the inertia of the elastic membrane, and that the presence of aerodynamic transmission routes can be neglected. But now we will more accurately describe the behavior of the membrane, namely, we will take into account that, in contrast to model (5), in reality, when the input signal $P(t)$ acts, each point of the membrane will have its own deviation.

If the membrane used in the MC has a rectangular shape, the dynamic properties of the considered MC are described by the boundary problem

$$k_1(t)\frac{\partial W(x,y,t)}{\partial t}+\rho_s\frac{\partial^2 W(x,y,t)}{\partial t^2}=$$

$$=\frac{1}{\lambda_0}\left[\frac{\partial^2 W(x,y,t)}{\partial x^2}+\frac{\partial^2 W(x,y,t)}{\partial y^2}\right]+P(t),\ t>0, \tag{16}$$

$$W(x,y,t)\big|_{\tilde{a}}=0,\ t\geq 0;\ W(x,y,0)=W_0;\ W_t'(x,y,0)=W_1, \tag{17}$$

where $W(x, y, t)$ is the instantaneous value of the deviation of the membrane point with coordinates (x, y) from the equilibrium position; $k_1(t)$ - the function characterizing the damping of the membrane; ρ_s - the mass of a unit area of the membrane; λ_0 - the value characterizing the material of which the membrane is made, as well as its elasticity and tension; the index "г" indicates the boundary of the membrane.

If the membrane used in the MC has a circular shape, the dynamic properties of the MC are described by the boundary problem

$$k_1(t)\frac{\partial W(r,t)}{\partial t}+\rho_s\frac{\partial^2 W(r,t)}{\partial t^2}=\frac{1}{\lambda_0}\left[\frac{\partial^2 W(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial W(r,t)}{\partial r}\right]+P(t), \tag{18}$$

$$W(R,t=0,\ t\geq 0;\ W(r,0)=W_0;\ W_t'(r,0)=W_1, \tag{19}$$

where r - the distance from the center of the membrane to a given point; R - the radius of the membrane.

Similarly, it is possible to clarify the behavior of aerodynamic transmission routes, describing them with the corresponding boundary problem.

These models of MC's are widely known and, of course, they do not exhaust either the huge variety of existing models of measuring converters, or their complexity.

1.3. SOURCES OF PARAMETRIC PHENOMENA

If in the above linear models the parameters characterizing the dynamic properties of the object are constant in time, then according to existing terminology, these objects are called *linear stationary*. Otherwise, these objects are called *linear non-stationary*. As part of the analysis tasks, the theory of linear stationary objects has been fully developed. The theory of linear non-stationary dynamic objects, which turned out to be much more complex, is still in the stage of development. This applies fully to the measurement area.

In this paper, the main attention is paid to the so-called parametric effects appearing in the behavior of linear non-stationary subsystems included in the structure of the measurement system. The parametric effect will be considered as the effect of the appearance in the readings of the measurement subsystems of components, conditioned only by the fact of time-variability of the parameters of this subsystem.

As noted earlier, parametric effects can appear both in subsystems of obtaining primary measurement information, and in subsystems of transformation of measurement information. Therefore, analyzing a particular mathematical model in the future, we will simply talk about a measuring system (MS), without specifying which subsystems we are talking about. This is natural, because the same mathematical model can describe the dynamic properties of both the first of these subsystems and the second.

Regardless of the specific structure of the model and the corresponding source of parametric phenomena, in all cases the same, main, question arises. That is, how in this situation to restore the measured physical value $X(t)$ according to the available readings $Y(t)$ of the measuring system and the given mathematical model of the MS.

The most traditional and easiest way to solve this problem is that, instead of a time-variable parameter, to conditionally designate it $a(t)$, and take some constant value a. The value of a is thus an estimate. Sometimes it is calculated on the basis of known patterns, if any, characterizing the dependence of this parameter on other known physical and technological parameters. An example of an analysis of the possibilities of such an assessment in a particular area of measurement will be given below, where it will be shown that these capabilities are very limited. Therefore, most often the constant value of a is determined experimentally on special installations and stands serving for the dynamic identification of measuring systems. At the indicated installations and stands, they always try to create such identification conditions under which the investigated MSs could be considered stationary systems, i.e., conditions that allow the determined parameter to be considered approximately constant in the identification process.

However, no matter how the constant parameter a is assessed - theoretically or experimentally - a fundamental question arises and remains open: to what extent the results of the recovery of the measured signal $X(t)$, based on the use of output signal $Y(t)$ and the MS model containing a constant parameter a would be adequate for objective results of the recovery of the measured signal $X(t)$, which would have to be based on the output signal $Y(t)$ and the MS model containing the actual variable $a(t)$.

The second way to solve this problem of recovery of the measured signal when using non-stationary MC's, the beginning of which was started by A. N. Gordov [4], is as follows: the corresponding equation describing the dynamic properties of the non-stationary MS is solved, which makes it possible to establish a functional relationship between the reading $Y(t)$ of the measuring system and the measured signal $X(t)$, which, as a solution to the corresponding equation, contains the variable parameter $a(t)$. Then, setting various assumed functional types of the parameter $a(t)$, for each of them, according to the existing solution, the deviation of the readings is estimated $Y(t)$ of the IC from the measured signal $X(t)$. On the basis of these theoretical estimates of the values of deviations between $Y(t)$ and $X(t)$ the possible errors of measurement in the real process of measurement are evaluated in the future.

Thus, on the described recovery path of the measured signal $X(t)$ for non-stationary MS's, just as in the previous case of stationary MS's, one of the important steps is the stage of using solutions to direct problems. Therefore, after the appearance of this work by A. N. Gordov, various authors have published several dozens of works, the only purpose of which was to obtain as accurate as possible solutions to direct problems for non-stationary MS. The paper [1] gives a brief overview of the methods for solving these problems and a detailed analysis of the considered second way of solving the problem of reconstructing the measured signal $X(t)$ for non-stationary MS's is carried out. This path is currently the main one, and continues to develop.

Let us turn now to the question of the adequacy of the measured signal being restored along this path to the true measured signal. It is obvious that a more substantive answer can now be given to the basic question than in the previous case. But such an answer requires setting more initial information, namely, knowledge of the types of supposed laws of variation of the variable parameter $a(t)$, and the values of the constant values included in these laws. Thus, in order to give an objective assessment of the second way of solving the main problem of the measured signal recovery for non-stationary MS's, it is necessary to find out, firstly, how realistic it is to obtain the mentioned additional initial information about the MS's parameters' variables, and secondly, how much this additional information obtained before the measurement will correspond to the actual nature of change of these parameters in the process of real measurement. In order to clarify these issues, it is natural to turn to a measurement area in which the relevant physical patterns already known in this area can be relied upon. Of the measurement area discussed above, this is the temperature measurement area.

In the models related to temperature measurement, it was stated above that the main source of parametric effects is the parameter $\alpha_k(t)$, called the convective heat transfer coefficient. We will turn to the information currently available on this parameter. This information is generally known [11] and is provided here only to illustrate the actual situation of non-stationary measurement systems.

From theory it is known that the value of convective heat transfer coefficient depends on many physical factors, and this dependence is so complex that the only reliable data here are the empirical results of research. The ratios corresponding to these empirical results are usually constructed in a criterion form. Necessary criteria for further use in convective heat transfer theory are the criteria of Nusselt, Prandtl and Reynolds. Let us give definitions of these criteria, assuming, for the sake of specificity, that we are talking about heat sensors with the form of an uncut cylinder.

The Nusselt criterion has the form

$$\mathrm{Nu} = \frac{\alpha_k d}{\bar{\lambda}},$$

where α_k - the convective heat transfer coefficient, which is actually a function of time $\alpha_k(t)$; $\bar{\lambda}$ – the coefficient of thermal conductivity of the medium in which the cylindrical body is located; d - determining the size of the body, which is taken as the diameter of the cylinder.

The Nusselt criterion characterizes the intensity of heat exchange between the heat sensor and the medium.

The Prandtl criterion has the form

$$\mathrm{Pr} = \frac{\bar{\nu}_0}{a^*},$$

where ν_0 and a^* are the coefficients of kinematic viscosity and temperature conductivity of the medium, respectively.

This criterion characterizes the physical properties of the medium the temperature of which is to be measured.

The Reynolds criterion has the form

$$\mathrm{Re} = \frac{vd}{\bar{\nu}_0},$$

where v is the flow rate of liquid and gas flows washing the heat sensor.

Reynolds's criterion characterizes heat transfer during forced convection, i.e. in conditions where the movement of liquid and gas is caused by the action of any external forces; these flows represent greatest interest.

Note that the values of these criteria depend on temperature, so it is usually indicated that the values of these criteria are calculated at a temperature equal to either the temperature of the body surface $u_{п}$, the ambient temperature θ, or the arithmetic mean of the two temperatures.

In the theory of convective heat transfer, it is shown that under conditions of laminar flow of liquid ($10 < \mathrm{Re} < 10^3$) the criterion ratio of convective heat exchange is as

$$\mathrm{Nu}_\theta = 0.5(\mathrm{Re}_\theta)^{0.5} \cdot (\mathrm{Pr}_\theta)^{0.38} \cdot \left(\frac{\mathrm{Pr}_\theta}{\mathrm{Pr}_{п}}\right)^{0.25},$$

and in turbulent current conditions ($10^3 \le \mathrm{Re}_\theta \le 2 \cdot 10^5$)) the corresponding ratio is

$$\mathrm{Nu}_\theta = 0.25(\mathrm{Re}_\theta)^{0.6} \cdot (\mathrm{Pr}_\theta)^{0.38} \cdot \left(\frac{\mathrm{Pr}_\theta}{\mathrm{Pr}_{п}}\right)^{0.25}.$$

Under conditions of developed turbulence the degree index of Reynolds criterion can reach a value of 0.8.

Supposing that changes in the Prandtl criterion can be neglected in the measurement process, then both criteria ratios can be written as (index θomitted for ease of writing)

$$\mathrm{Nu}_\theta = \bar{c}_0 \mathrm{Re}^n,$$

where the meaning of constant $\bar{c}_0$, n is obvious from the above criteria ratios for laminar and turbulent currents.

The Reynolds criterion contains the flow rate, which is a function of time and spatial coordinates. Since we are interested in the dependence of convective heat transfer coefficient $\alpha_k(t)$ on time, naturally, the question arises: what changes in the value of the convective heat transfer coefficient lead to changes in the value of the flow rate? To establish the relationship between the specified values, we present the criterion Re in the form $\mathrm{Re} = \mathrm{Re}_0 + \widetilde{\mathrm{Re}}$, where Re_0 and $\widetilde{\mathrm{Re}}$ – are the constant and variable components of this criterion. Further, decomposing the value of Re^n into a power series by the variable $\widetilde{\mathrm{Re}}$ and limited to the linear members of the decomposition, we get

$$\mathrm{Nu} = \mathrm{Nu}_0 + \mathrm{Nu}_0 \cdot n \frac{\widetilde{\mathrm{Re}}}{\mathrm{Re}_0}, \quad \mathrm{Nu}_0 = c\mathrm{Re}_0^n.$$

Now, given that

$$\frac{\mathrm{Nu} - \mathrm{Nu}_0}{\mathrm{Nu}_0} = \frac{\widetilde{\mathrm{Nu}}}{\mathrm{Nu}_0} = \frac{\tilde{\alpha}_k}{\alpha_{k0}}, \qquad \frac{\widetilde{\mathrm{Re}}}{\mathrm{Re}_0} = \frac{\tilde{v}(t)}{\bar{v}_0},$$

where Nu_0, $\widetilde{\mathrm{Nu}}$ - constant and variable components of Nu criterion, and $\bar{v}_0$ и $\tilde{v}(t)$ - constant and variable components of flow rate, finally we have

$$\frac{\tilde{\alpha}_k(t)}{\alpha_{k0}} = n \frac{\tilde{v}(t)}{\bar{v}_0}.$$

In obtaining this ratio, it was assumed that the thermal conductivity coefficient λ_0 and the kinematic viscosity coefficient of the medium $\bar{\nu}_0$, in which the heat sensor is located, remain constant during the measurement.

The latter ratio shows that the time variability of the convective heat transfer coefficient $\alpha_k(t)$ is very significant when measuring flow temperatures: the relative value of the variable component of this coefficient is directly proportional to the relative value of the variable component of the flow rate. It is very important that this variable of the $\alpha_k(t)$ parameter is a natural and inevitable consequence of the measurement method itself - measurement by immersion of the transducer into the medium, the temperature of which must be measured, and the physical nature of the object of study - fluid and gas flows.

Thus, referring to the latter ratio, we can only judge the nature of the change in the parameter $\alpha_k(t)$ during the flow temperature measurement to the extent that we can evaluate the nature of the change in time of the flow rate - another basic physical characteristic of the flow.

However, as in the field of flow velocity measurement, primary converters are also subject to parametric effects, others of natural origin and possibly more complex, our attempt to solve the problem of parametric effects in one area of measurement was limited to the fact that the problem was transferred to another area of measurement.

Thus, while the second way of recovering the measured signal $X(t)$ from the $Y(t)$ of the non-stationary MS is more meaningful than the former, it does not allow us to bring this content to specific functional relations that would objectively take into account the influence of parametric effects, and thus implement the possibility of improving the accuracy of measurement. However, the second way of recovery of the measured signal $X(t)$, based on the solution of a direct problem with variables over time parameters, is independent and important, since it is the analysis of the direct problem's solution that allows us to establish and study the very complex regularities that accompany the real measurement processes, namely, the regularities of parametric effects' appearance in the behavior of non-stationary measuring systems.

1.4. MANIFESTATION AND ANALYTICAL ACCOUNTING OF PARAMETRIC EFFECTS

Consider the examples of the implementation of the second recovery path of the measured signal $X(t)$ from the indications $Y(t)$ of a non-stationary MS, based on the solution of the equation with variables in time parameters describing the dynamic properties of the investigated MS. The functional relationship between the measured signal $X(t)$ and the $Y(t)$ of the MS, which is established as a result of solving this equation, will allow us to specify the content of the problems of parametric phenomena in the behavior of unsteady MS's.

Measuring system with lumped parameters

Let us consider the measuring system described by model (II) at $n = 1$, that is, the relationship between the input $X(t)$ and output $Y(t)$ signals is determined by the equation

$$\frac{dY(t)}{dt} + a(t)Y(t) = a(t)X(t), \quad Y(0) = 0, \tag{20}$$

where for ease of writing, the symbol "0" for parameter $a(t)$ is omitted in the original equation.

Here and in the future, in order to simplify the conclusions of the necessary formulas, initial conditions, as a rule, are accepted as zero.

Suppose the parameter $a(t)$ and the measured signal $X(t)$ are linear functions of time:

$$a(t) = a_0 + v_a \cdot t, \quad X(t) = X_0 + v_x \cdot t, \tag{21}$$

where constant values X_0, a_0 - initial values of input signal and parameter; constant values v_x, v_a - speed of change of input signal and parameter respectively.

The solution of equation (20) with zero initial condition has the form

$$Y(t) = \int_0^t a(\tau)X(\tau)\exp\left[-\int_\tau^t a(\eta)\,d\eta\right]d\tau. \tag{22}$$

Substituting (21) in (22) and performing all possible intermediate calculations, get the expression for the MS readings

$$Y(t) = X(t) - X_0 \cdot \exp\left[-a_0 \cdot t - \frac{v_a}{2}t^2\right] - v_x\sqrt{\frac{2}{v_a}} \cdot \exp\left[-\left(\sqrt{\frac{v_a}{2}}t + \frac{a_0}{\sqrt{2v_a}}\right)^2\right]\frac{\sqrt{\pi}}{2} \times$$

$$\times\left[Erfi\left(\sqrt{\frac{v_a}{2}}t + \frac{a_0}{\sqrt{2v_a}}\right) - Erfi\left(\frac{a_0}{\sqrt{2v_a}}\right)\right], \quad v_a > 0 \tag{23}$$

$$Y(t) = X(t) - X_0 \cdot \exp\left[-a_0 \cdot t - \frac{|v_a|}{2}t^2\right] -$$

$$v_x\sqrt{\frac{2}{|v_a|}} \cdot \exp\left[\left(\sqrt{\frac{|v_a|}{2}}t - \frac{a_0}{\sqrt{2|v_a|}}\right)^2\right]\frac{\sqrt{\pi}}{2} \times$$

$$\times\left[erf\left(\sqrt{\frac{|v_a|}{2}}t - \frac{a_0}{\sqrt{2|v_a|}}\right) + erf\left(\frac{a_0}{\sqrt{2|v_a|}}\right)\right], \quad v_a < 0, \tag{24}$$

where $Erfi\ z = \frac{2}{\sqrt{\pi}}\int_0^z e^{x^2}dx, \quad erf\ z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx.$

For the MS under consideration, let us recall the basic conclusion of the stationary MS theory for the case of the linearly variable input signal: if the input signal $X(t)$ varies linearly, then the output signal $Y(t)$ of the linear stationary system (a system for which $v_a = 0$, $a(t) \equiv a_0$) in the steady state also changes linearly; the rates of change of the input and output signals coincide, and the values of these signals are shifted relative to each other, by a constant value v_x/a_0. In other words, in steady state there is a ratio of

$$Y(t) = X(t) - \frac{v_x}{a_0} = X_0 + v_x t - \frac{v_x}{a_0}. \tag{25}$$

From the expressions (23) and (24), it is clear that if the input signal and parameter are linear functions of time, the formulated conclusion of the theory of stationary MS ceases to be fair: the non-stationary MS readings are complex time functions that differ significantly from the input signal.

For the measurement conditions under consideration, the deviation $Y(t)$ of the MS from the measured signal $X(t)$ will depend on the speed sign v_a of the parameter change: as the parameter's values increase ($v_a > 0$), the indicated deviation tends to zero over time, and when the parameter values decrease ($v_a < 0$), this deviation will increase in an unlimited fashion, and therefore, the measurement loses its meaning.

As we see, in order to use relations (23) and (24) in order to assess the degree of reliability of the measurement results, it is necessary to know the initial information, which can only be of a hypothetical nature.

In the framework of model (20), we proceed to the analysis of one of the most typical and important measurement modes for practice, namely, we will consider the measurement conditions under which the input signal $X(t)$ and parameter $a(t)$ change in time according to a harmonic law:

$$X(t) = X_0 + A_x \cdot \sin(\omega t + \varphi), \quad a(t) = a_0 + A_a \cdot \sin \omega t, \tag{26}$$

where X_0, a_0 и A_x, A_a - constant values, with the meaning of the mean values and amplitudes of the input signal and parameter respectively; ω- cyclic frequency of oscillations; φ – phase shift between oscillations $X(t)$ и $a(t)$.

Substitute (26) in (22) and use special decomposition to calculate the integral:

$$\exp(\eta \cos \omega t) = I_0(\eta) + 2\sum_{n=1}^{\infty} I_n(\eta) \cos n\omega t,$$

$$\exp(-\eta \cos \omega t) = I_0(\eta) + 2\sum_{n=1}^{\infty} (-1)^n I_n(\eta) \cos n\omega t,$$

where $\eta = \frac{A}{\omega}, I_n(\eta)$ - the modified Bessel function of the first kind of n-th order.

As a result of these calculations, we obtain a relationship that establishes the relationship between the MS and the measured signal. In steady mode, it has the form (for clarity it is given in an expanded form)

$$Y(t) = X(t) - A_x \omega \left\{ I_0^2(\eta) \cdot \frac{1}{a_0^2 + \omega^2} \cdot \left[\omega \sin(\omega t + \varphi) + a_0 \cos(\omega t + \varphi)\right] + \right.$$

$$+ I_0(\eta) \sum_{n=1}^{\infty} I_n(\eta) \left(\frac{\sin[(n+1)\omega t + \varphi] - \sin[(n-1)\omega t - \varphi]}{a_0^2 + \omega^2} \omega + \right.$$

$$\left. + \frac{1}{a_0^2 + \omega^2} \left[\cos[(n+1)\omega t + \varphi] + \cos[(n-1)\omega t - \varphi]\right] a_0 \right) +$$

$$+ I_0(\eta) \sum_{n=1}^{\infty} (-1)^n I_n(\eta) \left(\frac{(n+1)\omega \sin[(n+1)\omega t + \varphi] + a_0 \cos[(n+1)\omega t + \varphi]}{a_0^2 + (n+1)^2 \omega^2} + \right.$$

$$\left. + \frac{(n-1)\omega \sin[(n-1)\omega t - \varphi] + a_0 \cos[(n-1)\omega t - \varphi]}{a_0^2 + (n-1)^2 \omega^2} \right) +$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^n I_n(\eta) I_k(\eta) \left(\frac{\sin[(n+k+1)\omega t + \varphi] + \sin[(n-k+1)\omega t + \varphi]}{a_0^2 + (n+1)^2 \omega^2} (n+1)\omega + \right.$$

$$+\frac{\cos[(n+k+1)\omega t+\varphi]+\cos[(n-k+1)\omega t+\varphi]}{a_0^2+(n+1)^2\omega^2}a_0+$$

$$+\frac{\sin[(n+k-1)\omega t-\varphi]+\sin[(n-k-1)\omega t-\varphi]}{a_0^2+(n-1)^2\omega^2}(n-1)\omega+$$

$$+\left.\left.\frac{\cos[(n+k-1)\omega t-\varphi]+\cos[(n-k-1)\omega t-\varphi]}{a_0^2+(n-1)^2\omega^2}a_0\right)\right\} \quad (27)$$

Let us recall the two most important conclusions of the stationary MS theory applied to the model (20): if the input signal $X(t)$ changes according to the harmonic law (26), the output signal $Y(t)$ of the linear stationary system (i.e., at $A_a = 0$, $a(t) \equiv a_0$) in a steady state changes according to the harmonic to the same type and frequency as the input signal (but with a different amplitude and phase bias), and the mean levels of oscillation between the MS and the input signal are the same.

As follows from (27), in the harmonic oscillation of the input signal, the output signal of a non-stationary MS is subject to a very complex pattern: monochromatic oscillation of the input signal unfolds in the readings of non-stationary MS in the full spectrum. Further, as follows from (27), the second statement of the theory of a stationary MS, namely, the statement about the coincidence of average levels of oscillations of input and output signals, loses validity. Indeed, the following expression emerges from (27) to offset the mean levels of these signals

$$\left.\frac{X_{\text{cp}}-Y_{\text{cp}}}{A_x}\right|_{t\to\infty}=\left.\frac{X_0-Y_{\text{cp}}}{A_x}\right|_{t\to\infty}=\frac{A_a}{a_0}\left[2\sum_{n=1}^{\infty}(-1)^n I_n^2(\eta)\frac{n}{\eta^2+n^2\left(\frac{A_a}{a_0}\right)^2}-\right.$$

$$\left.\frac{1}{\eta}I_0(\eta)I_1(\eta)-2\sum_{n=1}^{\infty}(-1)^n I_n(\eta)I_{n-1}(\eta)\frac{\eta}{\eta^2+n^2\left(\frac{A_a}{a_0}\right)^2}\right]\cos\varphi-$$

$$2\left(\frac{A_a}{a_0}\right)^2\cdot\sum_{n=1}^{\infty}(-1)^n I_n^2(\eta)\,n^2\sin\varphi\frac{1}{\eta\left[\eta^2+n^2\left(\frac{A_a}{a_0}\right)^2\right]}. \quad (28)$$

At synchronous and inphase oscillations of the input signal and parameter, the value of the specified offset is determined by the expression

$$\left.\frac{X_{\text{cp}}-Y_{\text{cp}}}{A_x}\right|_{t\to\infty}=\frac{A_a}{a_0}\left[2\sum_{n=1}^{\infty}(-1)^n\cdot n\cdot I_n^2(\eta)\frac{1}{\eta^2+n^2\left(\frac{A_a}{a_0}\right)^2}-\right.$$

$$\left.\frac{1}{\eta}I_0(\eta)I_1(\eta)-2\sum_{n=1}^{\infty}(-1)^n I_n(\eta)I_{n-1}(\eta)\frac{\eta}{\eta^2+n^2\left(\frac{A_a}{a_0}\right)^2}\right]. \quad (29)$$

Finally, for $\eta \to 0$, i.e. at high frequencies of input signal and parameter, we have

$$\left.\frac{X_{\tilde{n}\eth}-Y_{\tilde{n}\eth}}{A_x}\right|_{t\to\infty}=-\frac{1}{2}\frac{A_a}{a_0}. \quad (30)$$

The offset value defined by this expression is the largest.

Measuring system with distributed parameters

Consider the first model of non-stationary MS with distributed parameters. It has the form (6) - (7). We will conduct a study of this model to establish the form of the analytical relationship between input and output signals.

As we will see, from this relationship the conclusion follows in an obvious way, that in this case there are the same parametric effects that were analyzed above, with respect to the non-stationary MS described by the model (20).

So, let the behavior of MS be described by the boundary problem:

$$\frac{\partial u(x,t)}{\partial t} = a_0^2 \frac{\partial^2 u(x,t)}{\partial x^2} + m(t)[\theta(t) - u(x,t)], \quad t > 0, \quad x \in (0, l), \tag{31}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad u(l,t) = u_{\text{ст}} = \text{const}, \quad u(x,0) = u_0 = \text{const}. \tag{32}$$

Here u $u(x, t)$ - the local response of the MS, $\theta(t)$ - the measured, i.e. the input signal; all other symbols have the meaning specified in paragraph 1.2.

Here, the main source of parametric effects is the time variability of the parameter $m(t)$. Without the loss of generality, in the sequel we set $u_0 = u_{\text{ст}}$. Let's replace $\upsilon(x, t) = u(x, t) - u_0$, then the problem will look like:

$$\frac{\partial \upsilon(x,t)}{\partial t} = a_0^2 \frac{\partial^2 \upsilon(x,t)}{\partial x^2} + m(t)[V_0(t) - \upsilon(x,t)], \quad V_0(t) = \theta(t) - u_0, \tag{33}$$

$$\frac{\partial \upsilon(0,t)}{\partial x} = 0, \quad \upsilon(l,t) = 0, \quad \upsilon(x,0) = 0. \tag{34}$$

One of the general methods of solving boundary value problems for non-stationary systems is the method of integral identities [9]. The integral identity, equivalent to the boundary problem (33) - (34), has the form

$$\left(\frac{\partial \upsilon(x,t)}{\partial t}, \eta\right) + a_0^2 \left(\frac{\partial \upsilon(x,t)}{\partial x}, \frac{\partial \eta}{\partial x}\right) + m(t)\left(\upsilon(x,t), \eta\right) = m(t) \cdot V_0(t)(1, \eta), \tag{35}$$

where $\eta = \eta(x)$ is an arbitrary continuously differentiable function in $(0, l)$, satisfying boundary conditions in (34), and $(f_1, f_2) = \int_0^l f_1(x) f_2(x) dx$ means a scalar product of functions $f_1(x)$ и $f_2(x)$.

Let the desired solution be determined by the amount of

$$\upsilon(x,t) = \sum_{i=1}^{N} a_i(t)\, \varphi_i(x), \tag{36}$$

where $\{\varphi_i(x)\}$ is a known system of coordinate functions; $\{a_i(t)\}$ is a system of functions to be defined.

Put the expression (36) in the identity (35) and take as $\eta(x)$ sequence $\varphi_k(x)$, $k = 1, 2, \ldots, N$. Then to determine $a_i(t)$, $i = 1, 2, \ldots, N$ we get a system of ordinary differential equations

$$\sum_{i=1}^{N} (\varphi_i, \varphi_k) \frac{da_i(t)}{dt} + \sum_{i=1}^{N} \left[a_0^2 (\varphi_i', \varphi_k') + m(t)(\varphi_i, \varphi_k)\right] a_i(t) =$$

$$= m(t)\, V_0(t)\, (1, \varphi_k), \quad k = 1, 2, \ldots, N. \tag{37}$$

As a system $\{\varphi_i(x)\}$ we use a system of orthogonal functions that meet boundary conditions:

$$\varphi_i(x) = \sin\, (2i - 1)\frac{\pi}{2l}(x + l).$$

Then to determine the functions $a_i(t)$ we get N unrelated differential equations

$$\frac{da_i(t)}{dt} + \left\{ a_0^2 \left[\frac{(2i-1)\pi}{2l}\right]^2 + m(t) \right\} a_i(t) = \frac{4}{\pi} \frac{1}{2i-1} m(t) V_0(t). \tag{38}$$

Solutions of equations (38) under zero initial conditions have the form

$$a_i(t) = \frac{4}{\pi} \frac{1}{2i-1} \int_0^t \exp\left\{ -a_0^2 \left[\frac{(2i-1)\pi}{2l}\right]^2 (t - \tau) - \int_\tau^t m(\varepsilon) d\varepsilon \right\} m(\tau) V_0(\tau) d\tau. \tag{39}$$

By introducing into consideration an infinite set of functions $\{\varphi_i(x)\}$, $\{a_i(t)\}$, we get the exact solution of the initial boundary problem

$$\upsilon(x,t) = \frac{4}{\pi} \sum_{i=1}^{\infty} \frac{1}{2i-1} \exp\left\{ -a_0^2 \left[\frac{(2i-1)\pi}{2l}\right]^2 \cdot t - \int_0^t m(\tau) d\tau \right\} \times$$

$$\times \int_0^t m(\tau)V_0(\tau)\exp\left\{a_0^2\left[\frac{(2i-1)\pi}{2l}\right]^2\cdot\tau+\int_0^\tau m(\varepsilon)d\varepsilon\right\}d\tau\cdot\sin\left[\frac{(2i-1)\pi}{2l}(x+l)\right]. \quad (40)$$

The solution (40) shows the local response of the MS to the signal $V_0(t)$. Get the expression for the integral reaction, i.e. for the MS readings. It has the form of

$$\upsilon(t)=-\frac{8}{\pi^2 l'}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^2}\cos\left[(2i-1)\frac{\pi}{2}(1+l')\right]\times$$

$$\times\int_0^t m(\tau)V_0(\tau)\exp\left\{-a_0^2\left[\frac{(2i-1)\pi}{2l}\right]^2(t-\tau)-\int_\tau^t m(\varepsilon)d\varepsilon\right\}d\tau,\quad l'=\frac{l_0}{l}. \quad (41)$$

If the output signal corresponds to the local response of the IC at $x = 0$, then

$$\upsilon(t)=\frac{4}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i-1}\cdot\int_0^t m(\tau)V_0(\tau)\exp\left\{-a_0^2\left[\frac{(2i-1)\pi}{2l}\right]^2(t-\tau)-\int_\tau^t m(\varepsilon)d\varepsilon\right\}d\tau\cdot \quad (42)$$

Comparing the equations (41) and (42) establishing the relationship between the input and output signals of the considered non-stationary MS with distributed parameters, with the expression (22) setting the relationship between input and output signals of a non-stationary MS with concentrated first - order parameters, it can be concluded that all the parametric effects discussed above in relation to non-stationary first order MS's are also present in the dynamics of the considered non-stationary MS with distributed parameters. Since the common members in the series (41) and (42) are the same in structure as the expression (22), these common members can be calculated for specific measurement modes in the same way as in the case of the expression (22).

In conclusion, we draw your attention to one feature, characteristic precisely for the considered MS with distributed parameters. Let the parameter $m(t)$ and the measured signal $\theta(t)$ remain constant during the measurement process. Then, assuming in the solution (40) $m(t) = m$ = const, $V_0(t) = V_0 = \theta - u_0$ = const, we get

$$\upsilon(x,t)=\frac{4mV_0}{\pi}\sum_{i=1}^{\infty}\frac{1}{2i-1}\sin\left[\frac{2i-1}{2l}\pi(x+l)\right]\frac{1}{a_0^2\left[\frac{(2i-1)\pi}{2l}\right]^2+m}\times$$

$$\times\left(1-\exp\left\{-a_0^2\left[\frac{(2i-1)\pi}{2l}\right]^2\cdot t-mt\right\}\right), \quad (43)$$

that for steady measurement mode it will take the form of

$$\upsilon(x,\infty)=\frac{4mV_0}{\pi}\sum_{i=1}^{\infty}\frac{1}{2i-1}\sin\left[\frac{2i-1}{2l}\pi(x+l)\right]\frac{1}{a_0^2\left[\frac{(2i-1)\pi}{2l}\right]^2+m}. \quad (44)$$

It follows from the expression (44) that even in steady state measurement, the local response of this stationary MS model does not match the measured signal θ at any spatial point. This obviously applies to the MS readings, regardless of how the MS readings are determined. Consequently, the parametric effects in these MS's are applied to the existing bias between the MS and the signal being measured. Since the specified offset has the nature of methodological error, the accounting of this offset is quite simple.

Similarly, the analysis of the behavior of non-stationary MS with distributed parameters described by the boundary problem (8) - (9) [1] is carried out.

CHAPTER 2

INVARIANCE IN THE DYNAMICS OF MEASURING SYSTEMS

This chapter discusses the statement of the problem of dynamic measurements and the content of physically single-channel invariance principle, the implementation of which allows us to exclude the influence of parametric effects in the dynamics of measurements.

2.1. ON THE FORMULATION OF THE PROBLEM OF DYNAMIC MEASUREMENTS

Let us continue the analysis of the possibilities of recovering the measured signal $X(t)$ when using a non-stationary MS, which was started in section 1.3. As follows from the analysis in p. 1.3, the specificity of the processes for measuring non-electric quantities is that the values of the parameters of the MS depend on the conditions under which the measurement will be carried out, and the most important of these conditions are determined not only by the properties of the MS itself, but often the properties of the object whose characteristic is to be measured. Note that preliminary estimates of the impact of such conditions on MS properties and measurement accuracy do not have significant value, since it is important to evaluate this influence in the process of actual measurement. Therefore, the generally accepted method described in p. 1.3 for reconstructing the measured signal $X(t)$ from the readings $Y(t)$ of unsteady MS's, and estimating the estimated error by solving the direct task, cannot be considered satisfactory: the estimation of the assumed error is based on the use of assumptions about unknown measurement conditions, including the properties of the object of study.

Let us refer to the existing formulation of the problem of dynamic measurements for linear MS's with lumped parameters.

Assume

$$AX(t) = Y(t),$$

where $X(t)$ – the signal to be measured; $Y(t)$ – the MS; A - the linear operator that fully describes the dynamic properties of the MS.

The task of dynamic measurements in general is formulated as follows: on the basis of the known MS $Y(t)$ and the given operator A to determine the measured signal $X(t)$.

Obviously, the problem of measurement is inherently the inverse problem, so in recent decades the problem of dynamic measurements is increasingly being solved as an inverse problem. The results obtained in this direction usually refer to a linear stationary MS, and are related to the construction of regularized solutions of inverse measurement problems, which generally speaking, refer to the class of so-called incorrectly assigned tasks. In the case of non-stationary MS's, problems associated with parametric phenomena are also fully encountered in solving the inverse problems.

So, in accordance with the given formulation of the problem of dynamic measurements, it is assumed that operator A is given; that is, both the structure of this operator and the numerical values of the values of the parameters that are included in the specified operator. In other words, the wording suggests that identification has already been achieved before the MS is used in the actual measurement process. This fact allows us to set the problem of determining the measured signal $X(t)$ in accordance with the above operator equation.

However, in connection with the specifics of the problems of measuring non-electric quantities noted at the beginning of this section, with the stated formulation and solution of the measurement problem, even for linear stationary MS's, an important question remains unanswered: the extent to which the values of stationary MS constant parameters determined during identification prior to the measurement process will correspond to the true values of these parameters in the actual measurement process. Without an answer to this question, it is impossible to judge whether the restored measured signal will be adequate in relation to the true measured signal. The situation is significantly aggravated by trying to cover the above dynamic measurement problem with a more general MS class, namely linear non-stationary MS's, which are the main subject of research in this paper.

From all the foregoing, the obvious conclusion is inevitable: the results of the recovery of the measured signal $X(t)$ shall be independent of the specific values of the parameters or functions characterizing the dynamic properties of the measuring system. In other words, the method (algorithm) of recovering the measured signal $X(t)$ must be invariant to the values of the specified parameters or functions. If the structure of the non-stationary MS model is known, invariance to the specific values of the magnitudes of unknown variables over time of the parameters included in the MS model structure is obviously required. If the structure of the non-stationary MS model is not known, assuming the above linearity condition of the MS is fulfilled, the main characteristic of the MS is, for example, its impulse transient function. Hence, in this case it is possible to approximate the unknown impulse transient function to some known function containing the linearly unknown parameters included in it, so the requirement of invariance to impulse transient function will mean invariance to specific values of magnitudes of specified unknown parameters.

Thus, bearing in mind only linear MS's, and in general, that is, in the case of unknown MS structure, invariance requirement can be interpreted as invariance of the method (algorithm) of recovery of the measured signal X(*t*) to the specific values of the magnitudes of unknown parameters characterizing the dynamic properties of the MS.

Note that the problems of finding a method (algorithm) of recovery of the measured signal, the results of which would not depend on specific values of magnitudes of unknown parameters characterizing dynamic properties of the measuring instruments used, are far from new. Moreover, measuring systems have been successfully operating in measuring technology for many decades, in the structure of which these methods are implemented - for example, in the field of measuring the angle of rotation and angular velocity of objects on which measuring instruments are installed, in the area of measurement of acceleration of objects moving in space, and others. One of the most successful and widespread methods turned out to be instrumental-algorithmic ones. The essence of these methods is that several measuring converters of primary measurement information ("sensors") are simultaneously used to measure a certain physical quantity - signal $X(t)$, and the recovery of the desired signal is carried out on the basis of using a special algorithm contained in its structure as readings $Y_i(t)$ of the used converters, as well as some operations on these readings.

The natural question arises: whether the specified instrumental-algorithmic method of recovering the measured signal is universal, and can this method be used, for example, in measuring physical characteristics of fluid and gas flows under such difficult conditions that accompany the measurement process in these objects of study.

To answer this question, it is necessary to refer to G. Pfrim [16], which is probably the first work in which an attempt was made to exclude the influence of parametric effects on the accuracy of dynamic measurements. The method of restoration of the measured signal proposed by Pfrim, can also be attributed to instrumental-algorithmic. In order to exclude the influence of parametric effects on the accuracy of dynamic temperature measurements of fluids and gases, Pfrim suggested using two measuring converters (two heat sensors), and the essence of the proposed method is as follows.

Let it be a question of measuring the temperatures of turbulent flows of liquids and gases, which is of the greatest interest. In this case, each flow point has its own temperature $X(t)$ and its own velocity $v(t)$ which are some functions of time; in general they are random time functions.

Let's make the first assumption. We will assume that at some two flow points where two heat receivers can be placed, the flow temperature turns out to be the same time function, i.e. $X_1(t) = X_2(t) = X(t)$. Let two simple heat sensors be used to measure the flow temperature (section 1.1). Then, taking into account the assumption made about the equality of flow temperatures at two different points, as mathematical models of these heat sensors can be taken:

$$\frac{dY_1(t)}{dt} + \frac{\alpha_1(t)}{c_1\gamma_1 L_1} Y_1(t) = \frac{\alpha_1(t)}{c_1\gamma_1 L_1} X(t), \quad L_1 = \frac{V_1}{S_1},$$

$$\frac{dY_2(t)}{dt} + \frac{\alpha_2(t)}{c_2\gamma_2 L_2} Y_2(t) = \frac{\alpha_2(t)}{c_2\gamma_2 L_2} X(t), \quad L_2 = \frac{V_2}{S_2}.$$

Here $X(t)$ – the measured flow temperature; $Y_1(t)$, $Y_2(t)$ – the readings of the first and second heat sensors, respectively, and all other designations, have a physical meaning as specified in paragraph 1.1.

As it is clear from the form of these equations, the source of parametric phenomena in the considered heat sensors are convective heat transfer coefficients $\alpha_1(t)$ and $\alpha_2(t)$, since all other physical parameters can be considered constant.

From these two equations we can find the flow temperature expressed through the values Y_1, Y_2, Y_1', Y_2' and the ratio of convective heat transfer coefficients:

$$X(t) = Y_1(t)\frac{1 - \frac{Y_2(t)}{Y_1(t)} \cdot f(t)}{1 - f(t)}, \quad \text{where} \quad f(t) = \frac{c_1\gamma_1 L_1}{c_2\gamma_2 L_2} \cdot \frac{\alpha_2(t)}{\alpha_1(t)} \cdot \frac{Y_1'(t)}{Y_2'(t)}.$$

Let us make a second assumption. We will assume that the flow rate at the two points where the heat sensors are located also turns out to be the same time function.

Finally, let us make the third assumption. We will assume that the measured flow temperature $X(t)$ change so little during the measurement that the criteria equations given in paragraph 1.3, which are valid for stationary convective heat exchange, can be used in the conditions of dynamic measurements, which are characterized by non-stationary convective heat exchange.

Then the ratio is fair:

$$\frac{\alpha_2(t)}{\alpha_1(t)} = \left(\frac{L_2}{L_1}\right)^{n-1},$$

where *n* - the exponent of the degree in the criterion equation (item 1.3).

Now for function *f*(*t*) we have

$$f(t) = \frac{c_1\gamma_1}{c_2\gamma_2}\left(\frac{L_1}{L_2}\right)^{2-n} \cdot \frac{Y_1'(t)}{Y_2'(t)}$$

and therefore the above flow temperature algorithm is independent of convective heat transfer coefficients $\alpha_1(t)$ and $\alpha_2(t)$.

Thus, despite the presence of parametric phenomena in the behavior of each of the heat sensors, under the assumptions made above, an algorithm for determining the measured signal from the readings of two heat sensors is independent of parametric phenomena.

It is easy to see that the method proposed by Pfrim to exclude the influence of parametric effects on the accuracy of dynamic measurements, in essence, can be considered as one of the many possible particular implementations of the ideas of the well-known theory of invariance, the structural sign of realizability of which is the so-called two-channel principle. As we know, the fact of emergence of the theory of invariance in the general and modern sense is noted in 1939, and is associated with the name of G. V. Schipanov.

Let us turn, however, to the substantive side of Pfrim's method, i.e. the degree of reliability of the assumptions made in the formulation of this method.

The first assumption – that the temperatures at the two flow points in which the heat receivers are located are the same function of time *X*(*t*), and the second assumption – that the velocity at the same two points of flow are the same function of time *v*(*t*), are in direct contradiction with the physical nature of the object under study: different points of turbulent flow of liquid and gas correspond to different temporal patterns of temperature and velocity changes.

The third assumption is also devoid of logical basis. Indeed, if criterion equations are used to produce the algorithm for the recovery of the measured temperature *X*(*t*) that are valid for stationary heat exchange with constancy or a slight change in *X*(*t*), this is not essentially an entirely dynamic measurement. If we are talking about really dynamic measurements, the very meaning of which is meant by the time variability of temperature *X*(*t*), and therefore, the manifestation of the laws of unsteady heat transfer, then using the criteria equations of stationary heat transfer is simply incorrect.

Thus, the assumptions discussed are so far from the real physical picture, which takes place in the dynamic measurements of temperatures of fluid and gas flows, that the built algorithm is essentially a reflection of some idealized situation of measurements. Although the *X*(*t*) signal recovery algorithm itself is accurate within the given assumptions, even minor deviations from these assumptions lead to unacceptable inaccuracies in the results. This is easy to verify by referring to the computer simulation of the recovery process of signal *X*(*t*) in accordance with the algorithm discussed. These circumstances have become an insurmountable obstacle to any noticeable practical application of the described method in measuring the temperature of fluids and gases flows.

However, the very important methodological importance of the work [16] should be noted: the author not only first drew the attention of researchers to the presence of a very important and complex problem in the theory of dynamic measurements - the problem of parametric phenomena. But he also pointed to one of the possible directions of the search for methods of solving this problem.

In view of the above, and returning to the question of methods of restoration of the measured signal, it is necessary to state: there are areas of measurement of non-electrical quantities, such as the area of measurement of the physical characteristics of fluids and gases, where the measurement conditions are so complex and specific that the use of instrumental-algorithmic ways to restore the measured signal in order to exclude the influence of parametric effects is not possible. All the above leads to the search for purely algorithmic ways to restore the measured signals, and such ones that would exclude the influence of parametric effects on the accuracy of dynamic measurements. At the same time, a purely algorithmic method of reconstructing the measured signal means that this method is based on the use of some special algorithms containing Y(*t*) readings of a given measuring system and mathematical operations on these readings, but does not provide for instrumental intervention in this MS, nor the use of any other additional measuring converters of primary measuring information.

Bearing in mind the earlier interpretation of the concept of invariance, let us assume that the dynamic properties of the SI are determined by a system of unknown parameters $\{a_i(t)\}$, $i = 0, 1, \ldots, (n-1)$, and all these parameters, and the unknown $X(t)$ signal to be measured, are independent components of some single system of unknown values, one unknown vector, and the measurement task is to simultaneously and independently determine each component of an unknown vector according to the readings of this MS $Y(t)$. The words "simultaneous and independent definition" in this case, means that all these unknowns are defined as independent solutions of some system of equations which contains in its structure the reading of the given $Y(t)$, and the mathematical operations on these readings.

The issues of construction of these systems will be discussed in detail in the future. Here, we will only assume the possibility of constructing some system of linear algebraic equations (SLAE), the solutions of which are either source unknown components $X(t)$, $\{a_i(t)\}$, $i = 0, 1, \ldots, (n-1)$, or some generalized (intermediate) components representing known conversions of source unknown components. The given statement of the problem of dynamic measurements can be considered as an extended task of measurement. We can assume that the extended measurement task is solvable, and pay attention to the consequences arising from this fact.

Since the initial unknown components (or intermediate components) are solutions of some SLAE, the values of the magnitude of each of the found initial components turn out to be invariant to the specific values of the magnitudes of all other components, and therefore, the values of the reconstructed signal $X(t)$ turn out to be invariant to specific values of the magnitudes of all parameters $\{a_i(t)\}$, $i = 0, 1, \ldots, (n-1)$ of the measuring system.

Further, if the extended measurement task is solved, that is, along with the measured signal $X(t)$ all unknown parameters $\{a_i(t)\}$, $i = 0, 1, \ldots, (n-1)$ are found, then the parametric identification problem is solved for linear non-stationary MS's, and the results of solving the problem of identification, firstly, are invariant to the specific values of the value of the input signal - the measured signal $X(t)$, and secondly, most importantly, the results of the identification problem refer to the actual measurement conditions.

Thus, the search for purely algorithmic ways to restore the measured signal $X(t)$, allowing us to exclude the influence of parametric effects on the measurement accuracy, led to the need to set dynamic dimension tasks as an extended task of dynamic measurements. The solution of the extended measurement task, in turn, means, in essence, the implementation of the physically single-channel principle of invariance in the dynamics of measurements.

Now, limited only to the scope of the problems involved in this work, and, above all, the problem of exclusion of the influence of parametric effects, the formulation of the problem of dynamic measurements can be formulated as follows: we determine the measured signal $Y(t)$ according to the readings $X(t)$ of the linear measuring system in accordance with the algorithm invariant to the characteristics of the dynamic properties of the given MS.

Note that the given wording does not require a statement of the operator characterizing the dynamic properties of the MS, but indicates the linearity of the MS. This means that in a special case, when the operator structure which characterizes the dynamic properties of the MS is defined and only the parameters of the MS included in this structure are not known, the algorithm of signal recovery $X(t)$ shall be invariant to the MS values.

In general, when both the structure of the operator characterizing the dynamic properties of the MS and the parameters of the MS included in this structure are not known, the algorithm of recovery of the signal $X(t)$ should be invariant to some generalized characteristic of the MS, for example, to its impulse transient function. As mentioned earlier, the unknown impulse transient function of the linear MS can be approximated by some known function containing the linearly unknown parameters. Therefore, invariance to an unknown impulse transient function will mean invariance to the values of unknown parameters included in the structure of the function approximating the impulse transient MS function.

It is obvious that the given formulation of the problem of dynamic measurements as an extended task of dynamic measurements is very conditional, has a private character, and any extension of the range of the considered measurement tasks, much less the class of measurement systems used, may require a substantial change in the content of this wording.

2.2. PHYSICALLY SINGLE-CHANNEL PRINCIPLE OF INVARIANCE

Let us refer to the detailed description of the algorithm of the physically single-channel invariance principle for linear non-stationary measuring systems with lumped parameters. Let us first touch on one methodological question related to the form of the mathematical model "input-output", taken to describe the behavior of linear dynamic systems. The specified mathematical model for a linear MS at zero initial conditions can have an "integral form" - the form of the Volterra integral equation of the first kind

$$\int_0^t g(t,\tau)X(\tau)d\tau = Y(t),$$

where $g(t, \tau)$ – the impulse transient function of the dynamic system; $X(t)$, $Y(t)$ – the input and output signals of the system, respectively.

The division of tasks into direct and inverse is connected with the relation of the content of the problem to the cause-effect relations: in direct tasks, the cause determines the effect. In reverse tasks the cause is restored by the effect. In the given model, the direct task has a single statement - it is the task of determining the output signal $Y(t)$ by the given impulse transient function $g(t, \tau)$ and the input signal $X(t)$.

The inverse task can have two productions. The first statement of the inverse task: by the given impulse transient function $g(t, \tau)$ and the output signal $Y(t)$ to determine the input signal of the system $X(t)$ – the statement of the problem of recovery of the measured signal.

The second statement of the inverse task: determine the impulse transient function $g(t, \tau)$ of the dynamic system by the given input $X(t)$ and output $Y(t)$ signals.

But the mathematical model "input-output" to describe the behavior of a linear dynamic system can have a different, for example, "differential form" - the form of an ordinary linear differential equation:

$$\frac{d^n Y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i(t) \frac{d^i Y(t)}{dt^i} = X(t), \quad Y^{(i)}(0) = Y_0^{(i)}, \; i = 0, 1, \ldots, (n-1),$$

in which the functions $Y(t)$, $X(t)$ have the same meaning as in the previous case, and instead of the generalized characteristic $g(t, \tau)$ of the dynamic system, the characteristic of that system is a linear differential operator containing variable parameters of the non-stationary measuring system.

All the above mentioned points about direct and inverse tasks apply, of course, to the second model. It is well known that for linear MS's, one of the mathematical models for the behavior of MS's can be switched to another. It is clear that in general the mathematical model "input-output" of the behavior of a linear dynamic system can take the form of some linear integration and differential equation.

Thus, the presented evidence that in setting direct and inverse problems of dynamic measurements the form of the applied mathematical model "input-output" does not have fundamental importance, but it is important to focus on the content of the task - to determine the investigation for the reason, or to determine the cause for the investigation. Moreover, as will be seen in the future, when solving for example the inverse task of recovering the measured signal, the task of only the mathematical model "input-output" is not enough.

In connection with the above, in this paragraph the following symbolic form will be used as a mathematical model "input-output" of linear dynamic MS behavior:

$$\Phi[Y(t), a_i(t), X(t)] = 0, \quad i = 0, 1, 2, \ldots, (n-1), \tag{1}$$

where $\{a_i(t)\}$, $i = 0, 1, 2, \ldots, (n-1)$ – a system of parameters characterizing the dynamic properties of the MS, which, like the measured signal $X(t)$, are deterministic functions of time.

The mathematical model is chosen in a symbolic form (1), instead of a more compact operator form, because the form (1) is more visible when presenting the contents of this paragraph. The structure of the expression for $\Phi[\ldots]$, depending on the type of the selected model form, may contain linear operators of differentiation and integration of functions $Y(t)$, $X(t)$. These operators are not specified in (1) in the explicit view with the aim of simplifying the recording. The symbolic form (1) can be considered as some equation, the desired function in which is determined only by the statement of the problem itself; hereinafter, it is only a question of solving the inverse task of reconstructing the measured signal. Regarding the structure $\Phi[\ldots]$ we shall consider the following conditions fulfilled: structure $\Phi[\ldots]$ is set and unchanged during the recovery of the measured signal $X(t)$, and all functions $Y(t)$, $a_i(t)$, $i = 0, 1, 2, \ldots, (n-1)$, $X(t)$ are included in this structure in a linear way. If the structure of the linear MS model is not known, it is possible to refer to the approximation of its impulse transient function, which is an exhaustive characteristic of the dynamic properties of the linear MS. Assuming the possibility of approximation of the impulse transient function by some known function containing the unknown parameters is indefinitely included in it, it can be assumed that the above conditions for a linear MS can always be performed.

Let us proceed with a sequential presentation of the various stages of action, the aggregate of which is a method of recovery of the measured signal $X(t)$ for a linear non-stationary MS - the method of which is the implementation of physically single-channel invariance principle in the dynamics of measurements.

1. Problem statement - transition to the extended task of dynamic measurements.

Assume that along with the unknown measured signal $X(t)$, all parameters $\{\{a_i(t)\}, i = 0, 1, 2, \ldots, (n-1)$, characterizing the dynamic properties of a linear MS are also unknown values. All unknown values $X(t)$, $\{a_i(t)\}$, $i = 0, 1, 2, \ldots, (n-1)$ will be considered as independent components of a single system of unknown values, i.e. unknown components of a vector. The measurement task will be the problem of independent and simultaneous determination of each component of the unknown vector $\{X(t), \{a_i(t)\}, i = 0, 1, 2, \ldots, (n-1)\}$. Obviously, the extended measurement task turns out to be much more complex than the usual measurement problem, in which only the measured $X(t)$ signal is unknown.

When expanding the dimension problem, the structure of the expression for $\Phi[\ldots]$ may retain a linear character relative to all unknown components included in it, but may be nonlinear relative to some of the specified components. As will be seen in the consideration of specific models, an example of maintaining linearity in the transition to an extended measurement task is the first general MS model with lumped parameters, the simplest example of linearity loss is the second general MS model with lumped parameters. Linearization in the extended measurement task is discussed below.

2. Approximation of the initial unknown parameters and the measured signal.

Let the extended measurement task be solved for a sufficiently narrow time change interval $t \in [t_1, t_1 + T0]$, in which t_1 - randomly selected but located close to the beginning of the measurement process time moment, and $T0$ - the length of this interval. Let us assume that all the required function components are smooth enough and can be approximated at the selected interval using known functions containing the unknown but constant values - that is, the:

$$a_i(t) = \varphi_i(t, b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}), \quad i = 0, 1, 2, \ldots, (n-1),$$

$$X(t) = \varphi_n(t, b_0^{(n)}, b_1^{(n)}, \ldots, b_m^{(n)}), \quad i = 0, 1, 2, \ldots, (n-1)$$

where φ_i, φ_n – known functions, and $b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}$, $i = 0, 1, 2, \ldots, n$ are unknown constants.

For ease of writing, the number $(m + 1)$ of unknown constants for all approximated functions is the same. Now equation (1) can be written in the form of

$$\Phi[Y(t), \varphi_i(t, b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)})] = 0, \quad i = 0, 1, 2, \ldots, n. \tag{2}$$

Thus, at this stage, the problem of finding the unknown component-functions $X(t)$, $a_i(t)$, $i = 0, 1, 2, \ldots, (n-1)$, based on equation (1) for the extended measurement task, is replaced by the problem of finding the constant values based on the equation (2). If the structure $\Phi[\ldots]$ is unknown and it is necessary to use the approximation of the impulse transient function of a known function with its linearly entering unknown constants, then, according to this step, it is necessary to approximate only the function $X(t)$ -measured signal.

3. Introduction of generalized (intermediate) unknown.

The purpose of the introduction of generalized (intermediate) unknown $\gamma_1, \gamma_2, \ldots, \gamma_q$, is to linearize the structure of expression for $\Phi[\ldots]$, if during the transition to an extended measurement task, the specified structure turned out to be nonlinear relative to some of the required components of the vector $\{\{a_i(t)\}, i = 0, 1, 2, \ldots, (n-1), X(t)\}$. The number q of these generalized unknowns is determined by the nature of the nonlinearities appearing during the transition to the extended measurement task. If the noted nonlinearity appeared, it will be transferred to equation (2). Therefore, linearization of the extended measurement task is carried out over equation (2), which after the introduction of the generalized unknown can be written as

$$\Phi[Y(t), b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}, \gamma_1, \gamma_2, \ldots, \gamma_q] = 0,$$

which lists both former unknowns, linearly included in the structure $\Phi[\ldots]$, and generalized unknowns.

Since now the structure $[\Phi[\ldots]$ is linear relative to all the unknowns included in it, then by entering a single notation of the unknown - $X_1, X_2, \ldots, X_p$, instead of the equation (2), we have

$$\Phi[Y(t), X_1, X_2, \ldots, X_p] = 0, \tag{3}$$

where p - the total number of unknown.

It is clear that a single designation of "unknown" is introduced only for convenience, and is not necessary.

If the linear MS structure is specified, the introduction of a generalized unknown usually does not cause any difficulties. If the general case is considered - for linear MS structure is not set, and there is a need to use impulse transient function and its approximation, then the linearization of the nonlinear structure of the equation (2) is possible, but presents some difficulties. In this case, some special transformations of the equation (2) are required before the introduction of the generalized unknown. This general case is discussed in detail in the fourth chapter.

4. Compilation and solution of the main SLAE.

Assuming the actions and conditions described in the previous paragraphs to determine the unknown $X_1, X_2, \ldots, X_p$, it is necessary to make a system of linear algebraic equations based only on equation (3). The construction of the specified SLAE can be carried out, if, for example, we use the ideas associated with the methods of solving direct boundary problems in numerical analysis. These ideas are often used in solving inverse problems; in particular, in solving problems of parametric identification. The general approach used to move to SLAE is as follows. Since in the process of solving the stated measurement problem, the true values of the unknowns are not found, but instead their estimates $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{p\,p}$, then the quantity

$$I\,[Y(t), \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_p] = \Phi\,[Y(t), \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_p],$$

which is commonly referred to as residual, is not equal to zero.

Demanding from a residual value the fulfillment of different conditions, we get different methods of the basic SLAE. Some of these methods are analyzed in detail in the next chapter. Here, as an example, we will touch on only one of them, which is essentially an analogue of the known collocation method.

According to this method, we require that the residual value is zero at points $t_1, t_2, \ldots, t_p = t_1 + T0$, the number of which is equal to the number of the unknown. This gives the desired SLAE in the form of

$$I\,[Y(t_k), \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_p] = 0, \quad k = 1, 2, \ldots, p \tag{4}$$

Following the decision of SLAE (4), it is necessary to revert to the estimates of the previous required constants $\tilde{b}_0^{(i)}, \tilde{b}_1^{(i)}, \ldots, \tilde{b}_m^{(i)}$, $i = 0, 1, 2, \ldots, n$, and then to the estimates of the initial desired functions $\tilde{X}(t)$, $\tilde{a}_i(t)$, $i = 0, 1, 2, \ldots, (n-1)$. Thus, each of the estimates of the sought unknown functions is in accordance with the algorithm invariant to the specific values of the estimates of all other functions, and consequently, the task of restoring the measured signal (determining the estimate of the measured signal) invariant to parametric effects is solved.

Now let us focus on some comments related to the stages of constructing the described method (algorithm) invariant to parametric effects.

A. Regarding the approximating functions $\varphi_i(t)$, $i = 0, 1, 2, \ldots, n$ we note that apart from the property of sufficient smoothness, the choice of these functions is unlimited, so it is natural to choose simple functions. In the practical implementation of the described algorithm it is often preferable to use algebraic polynomials of canonical form or trigonometric polynomials.

B. After solving the problem of estimation of unknown functions on the interval $[t_1, t_1 + T0]$, we can proceed to solving this problem at any other interval. In this case, the second interval can partially coincide with the first, be paired with it, or not have common points with the first interval, and the results of solving the problem at the second interval are not related to the results of solving the problem at the first interval.

C. An important condition for the legality of the use of the stated algorithm of recovery of the sought value is that in the process of recovery of the measured signal $X(t)$ on the specified interval, the dynamic properties of this MS should be fully manifested. This condition is a consequence of a more general requirement in the description of form (1); namely, the structure $\Phi[\ldots]$ is not only given and linear with respect to the functions $Y(t)$, $a_i(t)$, $X(t)$ but it is also invariable in the recovery of the measured signal $X(t)$. The importance of this condition is related to the fact that, depending on the measurement conditions and the nature of the change in time of the function $X(t)$, the measuring system can "enter" into such a "dynamically established stage" of measurements in which the relationship between the input $X(t)$ and output $Y(t)$ signals are described by a much simpler relation than the original equation (1). It is clear that in these stages of measurement, not all the properties and features of the behavior of the MS characterizing its dynamic properties are manifested, but only some of them. Therefore, applying the initial equation (1) at this stage of measurement is not appropriate. This fact will be illustrated and discussed in the next chapter in the model implementation of the single-channel invariance principle. Here, we will only notice that taking into account this fact is especially important in the monotonous change of the measured signal $X(t)$.

D. When solving this inverse problem, it is possible to have a set of solutions, so it is necessary to have a means of choosing from this set, in one sense or another, the most acceptable solution.

An acceptable solution can be chosen by direct and indirect criteria. The use of direct criteria for choosing an acceptable solution requires special study, which is addressed at the end of chapter 5. Indirect criteria will be used to simulate the described $X(t)$ signal recovery algorithm. Indirect criteria will be the criteria that firstly, contain in their structure only the readings $Y(t)$ of the MS, and possibly some linear operations with $Y(t)$, as well as the found estimates of $\tilde{X}(t)$, $\tilde{a}_i(t)$, $i = 0, 1, \ldots, (n-1)$ and, secondly, allow us to indirectly determine the degree of proximity of the found estimates to the estimated values.

When implementing the described algorithm for reconstructing the measured signal as a result of the solution, errors appear. The main sources of these, among others can be:

- unsuccessful selection of the type of approximating functions $\varphi_i(t, b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)})$, $i = 0, 1, 2, \ldots, n$, in particular, if these functions have the form of algebraic polynomials, then the degree of the selected polynomials is not high enough;

- if the structure $\Phi[\ldots]$ is not known and we have to refer to the pulse transient function and its approximation when constructing this structure, the main source of error of the measured signal recovery may be the error of approximation of the pulse transient function of the MS.

In this case, the only "tool" for quantifying the results of verification of each of the alternative measuring situations, as well as comparing these results with each other, are the above criteria. The general program for recovering the measured signal $X(t)$ from the recorded values of the output signal $Y(t)$ provides for the simultaneous verification of a set from the specified alternative measurement situations.

The choice of an acceptable solution to the problem requires the presentation of certain conditions to the value of the indirect criterion. For example, for the selected acceptable solution, the value of the indirect criterion should either be the smallest in comparison with the values of this criterion for the other possible solutions, or it should not exceed a certain specified value. Thus, for the considered inverse measurement problem to be solvable, it is not enough, in contrast to the solution of the direct task, to set only the mathematical model of the "input-output" of the linear system, but it is still necessary to put forward in one form or another certain conditions, which must satisfy the error of solution of the problem.

When building indirect criteria, we can use the concept of residuals, which can have the above or other specially selected form: as an indirect criterion, we can take both the value of the residual itself at some selected point, and some functionality that uses this residual.

Consideration of indirect criteria will be continued in the next chapter.

The above is the content of the method (algorithm) of recovery of deterministic measured signals, invariant to parametric effects, which is essentially the implementation of a physically single-channel principle of invariance in the dynamics of non-stationary measuring systems. It is quite obvious that the transition from the formulation of the usual measurement task to the formulation of the extended measurement task, and this is the most important stage in the implementation of the method of recovery of the signal being measured, there arises a number of important, traditional in the solution of inverse task of dynamics of measuring systems, theoretical issues that require special and systematic research on the basis of a more general methodology for setting the measurement task, and using a more complex research apparatus. Such questions are not addressed here, as the purpose of this work is to study only the fundamental possibility of such a statement of the measurement problem, which would allow excluding the influence of parametric effects in dynamic dimensions. Therefore, the content of all further presentation is aimed at achieving exactly this specific goal, using the minimum necessary and simple mathematical means.

Note that the need to develop purely algorithmic ways to restore the signal $X(t)$, and therefore to implement the physically single-channel principle of invariance was noted in the monograph [2], which also proposed a possible algorithm of signal characteristics recovery, which however, leads to the solution of a system of nonlinear algebraic equations.

The purpose of this work is to construct algorithms for the recovery of the measured signal, which, being invariant to parametric effects, would eventually lead to the need to solve only systems of linear algebraic equations, which significantly simplifies the practical use of these algorithms.

2.3. INVARIANCE ALGORITHMS FOR LINEAR MEASURING SYSTEMS WITH LUMPED PARAMETERS

Combining the first (1.I) and the second (1.II) common models for linear measuring systems with lumped parameters, we have

$$\frac{d^n Y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i(t) \frac{d^i Y(t)}{dt^i} = F(t), \quad Y^{(i)}(0) = Y_0^{(i)}, \quad i = 0, 1, \ldots, (n-1). \tag{5}$$

Here for the first generic model $F(t) = X(t)$, and for the second $F(t) = a_0(t)X(t)$. Sometimes, if the differential equation (5) has a very high order, it may be advisable to move from the differential form of the model (5) to the integral form: according to a well-known method, the differential equation (5) can be reduced to the integral equation Volterra of the second kind.

Limiting the model (5) and proceeding from the above content of the physically single-channel invariance principle, we will consider the construction of an invariance algorithm in relation to the first and the second common MS models with lumped parameters.

FIRST GENERAL MODEL

As the initial model "input-output" in this case we have the equation

$$\frac{d^n Y(t)}{dt^n}+\sum_{i=0}^{n-1} a_i(t)\frac{d^i Y(t)}{dt^i}-X(t)=0,\ \ Y^{(i)}(0)=Y_0^{(i)},\ \ i=0,1,\ldots,\ (n-1), \tag{6}$$

where $\{a_i(t)\}$, $i = 0, 1, \ldots, (n - 1)$ – MS variables; $X(t)$ – unknown measured signal.

1. In accordance with the first step, proceed to the extended measurement task: along with the measured value of $X(t)$, all the parameters $a_i(t)$ of the system are also included among the unknowns. In addition, we assume that by registering the output signal $Y(t)$, it is always possible to determine its derivatives included in (6). So, the functions to be defined

$$X(t),\ a_i(t),\ i = 0, 1, \ldots, (n - 1).$$

Note that the structure of the equation (6) is linear relative to the one sought, and retains its linearity after moving to the extended measurement task.

2. Assume at the selected interval $[t_1, t_1 + T0]$ the solution to the problem of measuring the $a_i(t)$, $X(t)$ function is approximated by algebraic polynomials of the canonical form:

$$a_i(t) = \varphi_i(t, b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}),\quad i = 0, 1, \ldots, (n - 1),$$

$$X(t) = \varphi_n(t, b_0^{(n)}, b_1^{(n)}, \ldots, b_m^{(n)}),$$

where $b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}$, $i = 0, 1, \ldots, n$ – new unknown constants whose number is equal to $(n + 1)(m + 1)$.

Now instead of (6) we have the equation

$$\frac{d^n Y(t)}{dt^n}+\sum_{i=0}^{n-1}[b_0^{(i)}+b_1^{(i)}\cdot t+\ldots+b_m^{(i)}\cdot t^m]\frac{d^i Y(t)}{dt^i}-$$

$$-[b_0^{(n)}+b_1^{(n)}\cdot t+\ldots+b_m^{(n)}\cdot t^m]=0. \tag{7}$$

3. Since the model (7) is linear relative to the sought $b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}$, $i = 0, 1, \ldots, n$, there is no need to introduce a generalized unknown.

4. If the collocation method idea is used to determine the unknown constants $b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}$, $i = 0, 1, \ldots, n$ then equation (7) must be satisfied in the number of points equal to number of unknown, i.e. in $(n + 1)(m + 1)$ points. This gives the following SLAE for the determination of these constant values.

$$\sum_{i=0}^{n-1}\left[b_0^{(i)}+b_1^{(i)}\cdot t_k+\ldots+b_m^{(i)}\cdot t_k^m\right]\cdot\left[\frac{d^i Y(t)}{dt^i}\right]_{t=t_k}-$$

$$\left[b_0^{(n)}+b_1^{(n)}\cdot t_k+\ldots+b_m^{(n)}\cdot t_k^m\right]=-\left[\frac{d^n Y(t)}{dt^n}\right]_{t=t_k}, \tag{8}$$

$k = 1, 2, \ldots, (n + 1)(m + 1)$, $t_k = t_1 + T\cdot(k - 1)$, $T=\frac{T0}{(n+1)(m+1)-1}$.

Since the system of algebraic equations (8) is linear relative to the desired values, its solution is not difficult. The solution of this system allows us to find the values $b_0^{(i)}, b_1^{(i)}, \ldots, b_m^{(i)}$, $i = 0, 1, \ldots, n$, and therefore find unknown functions $X(t)$, $a_i(t)$, $i = 0, 1, \ldots, (n - 1)$, which is the solution to the problem of dynamic measurements, that is, the definition of the measured signal $X(t)$ in compliance with an algorithm invariant to parametric effects.

Second order measuring system

As an example, consider the type of SLAE (8) for the second order MS. If unknown parameters $a_i(t)$ and the desired signal $X(t)$ are approximated by linear functions, we have:

$$m = 1, \quad a_0(t) = b_0^{(0)} + b_1^{(0)} \cdot t, \quad a_1(t) = b_0^{(1)} + b_1^{(1)} \cdot t, \quad X(t) = b_0^{(2)} + b_1^{(2)} \cdot t.$$

When $m = 1$, $n = 2$, the general system (8) takes the form of

$$\left[b_0^{(0)} + b_1^{(0)} \cdot t_k\right] \cdot Y(t_k) + \left[b_0^{(1)} + b_1^{(1)} \cdot t_k\right] \cdot \left[\frac{dY(t)}{dt}\right]\Bigg|_{t=t_k} -$$

$$\left[b_0^{(2)} + b_1^{(2)} \cdot t_k\right] = -\left[\frac{d^2Y(t)}{dt^2}\right]\Bigg|_{t=t_k}, \qquad (9)$$

$k = 1, \ldots, 6$, $t_k = t_1 + T(k - 1)$, $T = T0/5$.

The single designation of the unknown here is obvious:

$$X_1 = b_0^{(0)}, \; X_2 = b_1^{(0)}, \; X_3 = b_0^{(1)}, \; X_4 = b_1^{(1)}, \; X_5 = b_0^{(2)}, \; X_6 = b_1^{(2)}.$$

SECOND GENERAL MODEL

As the initial input to output model (1), in this case we have

$$\frac{d^nY(t)}{dt^n} + \sum_{i=0}^{n-1} a_i(t)\frac{d^iY(t)}{dt^i} - a_0(t)X(t) = 0, \quad Y^{(i)}(0) = Y_0^{(i)}, \quad i = 0, 1, 2, \ldots, (n-1) \qquad (10)$$

Since the sequence and content of actions at the first two stages of the invariance algorithm's construction are the same as for the first general model, let us write immediately the equation obtained after approximation parameters $a_i(t)$ of the system and the measured signal $X(t)$:

$$\frac{d^nY(t)}{dt^n} + \sum_{i=0}^{n-1}\left[b_0^{(i)} + b_1^{(i)} \cdot t + \ldots + b_m^{(i)} \cdot t^m\right] \cdot \frac{d^iY(t)}{dt^i} -$$

$$[b_0^{(0)} + b_1^{(0)} \cdot t + \ldots + b_m^{(0)} \cdot t^m] \cdot [b_0^{(n)} + b_1^{(n)} \cdot t + \ldots + b_m^{(n)} \cdot t^m] = 0,$$

$$Y^{(i)}(0) = Y_0^{(i)}, \quad i = 0, 1, \ldots, (n-1). \qquad (11)$$

As we can see, the transition to the extended measurement task led to the fact that the model (11) was nonlinear relative to the search, which is due to the presence of products of unknown quantities in the equation (11).

For this model of measuring systems, the stage of linearization is the most important, so let us focus on it in detail.

In order for equation (11) to be linear with respect to unknown values, it is necessary to introduce generalized (intermediate) unknown values γ_1, γ_2, …, each of which represents one or another combination of the former unknown. Since the number of generalized unknowns depends directly on the degree of approximation of the polynomials, let us consider two particular cases for specificity: approximating polynomials are polynomials of the first and second degree.

A. Approximating polynomials are polynomials of the first degree.

In the case of $m = 1$, the expression (11) takes the form of

$$\frac{d^nY(t)}{dt^n} + \sum_{i=0}^{n-1}\left[b_0^{(i)} + b_1^{(i)} \cdot t\right] \cdot \frac{d^iY(t)}{dt^i} -$$

$$\left\{b_0^{(0)}b_0^{(n)} + [b_0^{(0)}b_1^{(n)} + b_1^{(0)}b_0^{(n)}] \cdot t + b_1^{(0)}b_1^{(n)} \cdot t^2\right\} = 0. \qquad (12)$$

As we can see, for the linearization of the model (12) it is necessary to introduce a generalized unknown:

$$\gamma_1 = b_0^{(0)}\, b_0^{(n)}, \quad \gamma_2 = b_0^{(0)}\, b_1^{(n)} + b_1^{(0)}\, b_0^{(n)}, \quad \gamma_3 = b_1^{(0)}\, b_1^{(n)}.$$

The number of the unknown contained in the second member of the left side (12) is $n\,(m + 1)$, hence the total number of unknown after linearization is $n\,(m + 1) + 3 = 2n + 3 = 2(n + 1) + 1$. For the first generic model, the total number of unknown in this case would be $(n + 1)\,(m + 1) = 2n + 2 = 2(n + 1)$.

B. Approximating polynomials are polynomials of the second degree.

In this case m = 2 and the expression (11) takes the form of

$$\frac{d^n Y(t)}{dt^n} + \sum_{i=0}^{n-1}\left[b_0^{(i)} + b_1^{(i)}\cdot t + b_2^{(i)}\cdot t^2\right]\cdot\frac{d^i Y(t)}{dt^i} -$$

$$\Big\{b_0^{(0)}b_0^{(n)} + [b_0^{(0)}b_1^{(n)} + b_1^{(0)}b_0^{(n)}]\cdot t + [b_0^{(0)}b_2^{(n)} + b_1^{(0)}b_1^{(n)} + b_2^{(0)}b_0^{(n)}]\cdot t^2 +$$

$$+ [b_1^{(0)}b_2^{(n)} + b_2^{(0)}b_1^{(n)}]\cdot t^3 + [b_2^{(0)}b_2^{(n)}]\cdot t^4\Big\} = 0 \qquad (13)$$

For linearization of the model (13) we introduce generalized unknown:

$$\gamma_1 = b_0^{(0)}b_0^{(n)},\quad \gamma_2 = b_0^{(0)}b_1^{(n)} + b_1^{(0)}b_0^{(n)},\quad \gamma_3 = b_0^{(0)}b_2^{(n)} + b_1^{(0)}b_1^{(n)} + b_2^{(0)}b_0^{(n)},$$

$$\gamma_4 = b_1^{(0)}b_2^{(n)} + b_2^{(0)}b_1^{(n)},\quad \gamma_5 = b_2^{(0)}b_2^{(n)}.$$

The total number of unknown after linearization is n (m + 1) + + 5 = 3n + 5 = 3(n + 1) + 2; for the first general model, the total number of unknown in this case would be 3 (n + 1).

Turning to the fourth stage of invariance algorithm construction, for the model (12) we get the following basic SLAE

$$\sum_{i=0}^{n-1}\left[b_0^{(i)} + b_1^{(i)}\cdot t_k\right]\cdot\left[\frac{d^i Y(t)}{dt^i}\right]\Bigg|_{t=t_k} -$$

$$\left[\gamma_1 + \gamma_2\cdot t_k + \gamma_3\cdot t_k^2\right] = -\left[\frac{d^n Y(t)}{dt^n}\right]\Bigg|_{t=t_k}, \qquad (14)$$

k = 1, 2, …, [2(n + 1) + 1], $t_k = t_1 + T(k-1)$, $T = \frac{T0}{2(n+1)}$.

The solution of system (14) gives the values of $b_0^{(i)}$, $b_1^{(i)}$, characterizing the parameters of the MS $a_i(t)$, and the values of the generalized unknowns γ_1, γ_2, γ_3. It remains to find the values $b_0^{(n)}$, $b_1^{(n)}$, characterizing the measured signal. Using the above expressions for γ_1, γ_2, γ_3 we have:

$$b_0^{(n)} = \frac{\gamma_1}{b_0^{(0)}},\quad b_1^{(n)} = \frac{\gamma_2 - b_1^{(0)}b_0^{(n)}}{b_0^{(0)}} \quad \text{èëè} \quad b_1^{(n)} = \frac{\gamma_3}{b_1^{(0)}}.$$

The values of $b_1^{(n)}$, determined by the two formulas are naturally identical.

Substituting the found values in approximating functions, we find estimates $\tilde{a}_i(t)$, $\tilde{X}(t)$, which completes the solution to the problem of restoring the measured signal.

The basic SLAE for the model (13) has the form

$$\sum_{i=0}^{n-1}\left[b_0^{(i)} + b_1^{(i)}t_k + b_2^{(i)}\cdot t_k^2\right]\cdot\left[\frac{d^i Y(t)}{dt^i}\right]\Bigg|_{t=t_k} -$$

$$\left[\gamma_1 + \gamma_2\cdot t_k + \gamma_3\cdot t_k^2 + \gamma_4\cdot t_k^3 + \gamma_5\cdot t_k^4\right] = -\left[\frac{d^n Y(t)}{dt^n}\right]\Bigg|_{t=t_k}, \qquad (15)$$

k = 1, 2, …, [3(n + 1) + 2], $t_k = t_1 + T(k-1)$, $T = \frac{T0}{3(n+1)+1}$.

The solution of the system (15) gives values of magnitudes $b_0^{(i)}$, $b_1^{(i)}$, $b_2^{(i)}$, i = 0, 1, 2, …, (n – 1) and values of generalized unknowns γ_1, …, γ_5. To find the values $b_0^{(n)}$, $b_1^{(n)}$, $b_2^{(n)}$, we will use expressions for generalized unknowns γ_1, γ_2 and, for example, γ_3. This gives:

$$b_0^{(n)} = \frac{\gamma_1}{b_0^{(0)}},\quad b_1^{(n)} = \frac{\gamma_2 - b_1^{(0)}b_0^{(n)}}{b_0^{(0)}},\quad b_2^{(n)} = \frac{\gamma_3 - b_1^{(0)}b_1^{(n)} - b_2^{(0)}b_0^{(n)}}{b_0^{(0)}}.$$

Substituting the found values of values $b_0^{(i)}$, $b_1^{(i)}$, $b_2^{(i)}$, i = 0, 1, 2, …, n in approximating functions, we get the required estimates $\tilde{a}_i(t)$, $\tilde{X}(t)$, which completes the solution of the measurement problem.

In conclusion, let us note that the introduction to the consideration of the generalized unknown leads to an increase in the order of the basic SLAE and the appearance of additional unknown, the found values of which may not be to be used.

2.4. INVARIANCE ALGORITHM FOR ONE CLASS OF NON-LINEAR MEASURING SYSTEMS

When considering the classical statement of the problem of dynamic measurements in p. 1.3, it was stated that the traditional approach to solving the problem of dynamic measurements consists of two stages: at the first stage the direct task of analysis is solved, and at the second stage - on the basis of the obtained solution of the direct task and the known readings $Y(t)$ of the MS, the measured signal $X(t)$ is evaluated, ascribing to the parameters of the MS some assumed values. As noted, even for linear MS's with deterministic changes in the input signal and MS parameters, difficulties arise already in the first stage in finding effective methods for solving direct tasks. In the transition to nonlinear measuring systems, the noted difficulties develop into a well-known problem of solving nonlinear differential and integral equations. In this regard, the need to build algorithms that are invariant to the parametric effects in nonlinear MS's is completely obvious, since the construction of these algorithms does not require the solution of any differential or integral equations.

Consider a class of nonlinear measuring systems, the behavior of which is described by the equation:

$$\sum_{i=0}^{n} g_i(Y)\frac{d^iY(t)}{dt^i} = X(t), \quad Y^{(i)}(0) = Y_0^{(i)}, \quad i = 0, 1, 2, ..., (n-1), \tag{16}$$

where, as before, $X(t)$, $Y(t)$ – input (measured) and output signals respectively.

This is a fairly typical model for a nonlinear MS with lumped parameters.

Let's assume that $g_i(Y)$ functions have form:

$$g_i(Y) = b_0^{(i)} + b_1^{(i)}Y + b_2^{(i)}Y^2 + ... + b_{m_i}^{(i)}Y^{m_i}, \quad i = 0, 1, 2, ..., n, \tag{17}$$

where $b_0^{(i)}, b_1^{(i)}, \dots, b_{m_i}^{(i)}$ – some constant values.

In this case, the main goal is to construct an algorithm for the recovery of the measured signal $X(t)$, which would be invariant to the values of all the constant values included in the structure of functions $g_i(Y)$. At the same time, let us assume that the structures of the equation (16) and the expressions (17) remain unchanged during the recovery of the measured signal. Turning to an extended measurement task in this case will mean that when constructing an invariance algorithm, along with the unknown measured signal $X(t)$, unknown and therefore subject to definition will also be considered all constant values $b_0^{(i)}, b_1^{(i)}, \dots, b_{m_i}^{(i)}$, $i = 0, 1, \dots, n$.

If at least one of the functions $g_i(Y)$ does not contain unknown constants, that is, a known constant value, or a known time function, or finally a known function of the output signal $Y(t)$, and neither the known time function nor the known output function contain unknown constant values, it is possible to construct a linear algorithm of recovery of the measured signals invariant to the values of unknown constants $b_0^{(i)}, b_1^{(i)}, b_2^{(i)}, \dots, b_{m_i}^{(i)}$ and based on the corresponding heterogeneous SLAE. In other words, the construction of an invariance algorithm is possible if, for at least one value of i, for example, for $i = p$, from the number of unknown constants that fall out, as known, are the constant values $b_0^{(p)}, b_1^{(p)}, \dots, b_{m_p}^{(p)}$, or even one of these constant values.

Consider only two of the many specified cases when the construction of an invariance algorithm is possible: the first case – $g_n(Y) = 1$, the second case – $g_0(Y) = 1$. Since the nature of the construction of the invariance algorithm is uniform for different orders n of equation (16), differing only in the amount of computation, we will restrict ourselves to a second-order system in order to simplify the further discussion.

Case $g_n(Y) = 1$

So assume $n = 2$ и $g_n(Y) = 1$, then the behavior of the measuring system is described by the equation:

$$\frac{d^2Y(t)}{dt^2} + g_1(Y)\frac{dY(t)}{dt} + g_0(Y)\cdot Y(t) = X(t), \quad Y(0) = Y_0, \quad Y'(0) = Y_0', \tag{18}$$

where $g_0(Y) = b_0^{(0)} + b_1^{(0)}Y + ... + b_{m_0}^{(0)}Y^{m_0}$; $g_1(Y) = b_0^{(1)} + b_1^{(1)}Y + ... + b_{m_1}^{(1)}Y^{m_1}$.

We will build the desired invariance algorithm in accordance with the steps set out in p. 2.2.

Moving to the extended measurement task, fix the unknown:

$$b_0^{(0)},\ b_1^{(0)}, ...,\ b_{m_0}^{(0)},\ b_0^{(1)},\ b_1^{(1)}, ...,\ b_{m_1}^{(1)},\ X(t).$$

Only the measured signal $X(t)$ should be approximated for the non-linear MS model under consideration. If the $X(t)$ function is approximated, for example, by an algebraic polynomial of the second degree,

$$X(t) = b_0^{(2)} + b_1^{(2)}t + b_2^{(2)}t^2,$$

then we have the following unknown constant values:

$$b_0^{(0)},\ b_1^{(0)}, ...,\ b_{m_0}^{(0)},\ \ b_0^{(1)},\ b_1^{(1)}, ...,\ b_{m_1}^{(1)},\ \ b_0^{(2)},\ b_1^{(2)}, b_2^{(2)},$$

the total number of which is $(m_0 + m_1) + 5$.

Since the model of the extended measurement task is linear with respect to all specified unknown constant values, for the considered model of the nonlinear MS, there is no necessity to introduce a generalized unknown.

Moving on to the construction of the basic SLAE, let us again use the idea of the collocation method. Then the main SLAE will take the form of

$$\left[b_0^{(1)} + b_1^{(1)}Y(t_k) + b_2^{(1)}Y^2(t_k) + ... + b_{m_1}^{(1)}Y^{m_1}(t_k)\right]\cdot\left[\frac{dY(t)}{dt}\right]\Bigg|_{t=t_k} +$$

$$+\left[b_0^{(0)} + b_1^{(0)}Y(t_k) + b_2^{(0)}Y^2(t_k) + ... + b_{m_0}^{(0)}Y^{m_0}(t_k)\right]\cdot Y(t_k) -$$

$$+\left[b_0^{(0)} + b_1^{(0)}Y(t_k) + b_2^{(0)}Y^2(t_k) + ... + b_{m_0}^{(0)}Y^{m_0}(t_k)\right]\cdot Y(t_k) - \qquad (19)$$

$$k = 1, 2, \ldots, [(m_0 + m_1) + 5],\ \ t_k = t_1 + T(k-1),\ T = \frac{T0}{(m_0+m_1)+4}.$$

Here t_1 – the starting point of recovery of the measured signal, $T0$ - the duration of the recovery interval.

The solution to this system gives all the desired values, and therefore allows us to restore the measured signal $X(t)$ at the interval $[t_1,\ t_1 + T0]$. The algorithm of recovery of the measured signal is invariant to the values of parameters $b_0^{(0)}, b_1^{(0)}, \ldots, b_{m_0}^{(0)},\ b_0^{(1)}, b_1^{(1)}, \ldots, b_{m_1}^{(1)}$ of nonlinear MS.

Note that along with the solution to the main measurement problem - the problem of the recovery of the measured signal $X(t)$, here, as before, the problem of identification is solved, namely, the problem of parametric identification of the non-linear MS in question.

Case $g_0(Y) = 1$

So, assuming $n = 2$ и $g_0(Y) = 1$, then the behavior of the measuring system is described by the equation

$$g_2(Y)\frac{d^2Y(t)}{dt^2} + g_1(Y)\frac{dY(t)}{dt} + Y(t) = X(t),\quad Y(0) = Y_0,\quad Y'(0) = Y_0', \qquad (20)$$

where: $g_1(Y) = b_0^{(1)} + b_1^{(1)}Y + ... + b_{m_1}^{(1)}Y^{m_1};\quad g_2(Y) = b_0^{(2)} + b_1^{(2)}Y + ... + b_{m_2}^{(2)}Y^{m_2}.$

Now in the extended measurement task, the unknowns are:

$$b_0^{(1)},\ b_1^{(1)}, ...,\ b_{m_1}^{(1)},\ b_0^{(2)},\ b_1^{(2)}, ...,\ b_{m_2}^{(2)},\ X(t).$$

If we approximate the measured signal with the algebraic polynomial of the second degree,

$$X(t) = b_0^{(3)} + b_1^{(3)}t + b_2^{(3)}t^2,$$

then the unknown constant values become:

$$b_0^{(1)},\ b_1^{(1)}, ...,\ b_{m_1}^{(1)},\ \ b_0^{(2)},\ b_1^{(2)}, ...,\ b_{m_2}^{(2)},\ \ b_0^{(3)},\ b_1^{(3)},\ b_2^{(3)},$$

the total number of which is $(m_1 + m_2) + 5$.

Due to the linearity of the extended measurement task relative to all given unknown constants, generalized unknowns are not entered.

The basic SLAE in this case has the form:

$$\left[b_0^{(2)} + b_1^{(2)} Y(t_k) + \ldots + b_{m_2}^{(2)} Y^{m_2}(t_k)\right] \cdot \left[\frac{d^2 Y(t)}{dt^2}\right]\Bigg|_{t=t_k} +$$

$$+\left[b_0^{(1)} + b_1^{(1)} Y(t_k) + \ldots + b_{m_1}^{(1)} Y^{m_1}(t_k)\right] \cdot \left[\frac{dY(t)}{dt}\right]\Bigg|_{t=t_k} -$$

$$\left[b_0^{(3)} + b_1^{(3)} t_k + b_2^{(3)} t_k^2\right] = -Y(t_k).$$

The solution to this system allows us to restore the measured signal $X(t)$ in accordance with the algorithm invariant to the values of parameters of the considered nonlinear MS.

In conclusion, we note that all of the above invariance algorithms were constructed in such a way that they were invariant also to the initial conditions, as in the considered measurement tasks, these conditions are often unknown.

CHAPTER 3

MODEL IMPLEMENTATION OF THE SINGLE-CHANNEL INVARIANCE PRINCIPLE

Upon stating the content of the invariance principle which is single-channel from a physical standpoint in the dynamics of non-stationary measurement systems, there arises a natural question regarding the possible verification of both the feasibility of this principle and the conclusions regarding the dynamic measurement theory, which are the consequences of this principle's application.

Since the arrangement of any physical experiment which would enable us to research such complex phenomena as parametric effects in the dynamics of non-electric quantity measurement is currently not possible, computer simulation is the only means of studying these issues.

This chapter considers in detail all the issues briefly addressed in the previous chapter when stating the content of the invariance principle, which is single-channel from a physical standpoint. Herewith, the basic purpose of the simulation is to confirm the fact that practical implementation of invariance algorithms, firstly, is not associated with overcoming any difficulties, and secondly, actually solves the problem of excluding the impact of parametric effects on the dynamic measurement's degree of accuracy.

The following simulation outcomes relate to the first-order measurement systems with deterministic changes in input signals and MS parameters. Herewith, more attention is paid to consideration of the mathematical model of this particular case, which corresponds to the second MS general model with lumped parameters (1.II), since the mathematical model for the first-order MS, resulting from the first MS general model with lumped parameters (1.I), is simpler both in terms of theoretical research and in terms of the implementation of the single-channel invariance principle. As for the higher-order measurement systems, the core of the subject does not change, although the computational procedures are without doubt more cumbersome.

In this chapter the actual (true) output signal of the measurement system which is produced at the actual (true) input signal $X(t)$ shall be denoted as $u(t)$; $u(t)$ signal shall be considered a known time-varying function, and $X(t)$ – an unknown time-varying function. Estimates of the measured signal $X(t)$ and the unknown MS parameter $a(t)$, which are produced as a result of the invariance algorithm implementation shall be denoted by $\tilde{X}(t)$, $\tilde{\alpha}(t)$.

3.1. METHODS OF CONSTRUCTING THE BASIC SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS FOR MEASURED SIGNAL RECOVERY

Construction of the basic simultaneous linear algebraic equations (SLAE) for restoring the measured signal parameters is a very important step in the invariance principle's implementation. Therefore, this step is considered in detail below.

Suppose the measurement system is described by the equation

$$\frac{du(t)}{dt} + a(t)u(t) = a(t)X(t), \quad u(0) = 0. \tag{1}$$

Based on the physical meaning of $a(t)$ parameter, we can consider as $a(t) \neq 0$. This parameter's equality to zero would mean that the system has an infinitely large inertia, and therefore cannot be used as a means of measurement. As stated above, this allows us to pass from the equation in the (1) form, to the equation in the form

$$\alpha(t)\frac{du(t)}{dt} + u(t) = X(t), \quad u(0) = 0, \; \alpha(t) = \frac{1}{a(t)}, \tag{2}$$

which seems to be more suitable for our immediate purposes.

Suppose the estimates $\tilde{\alpha}(t)$, $\tilde{X}(t)$ for unknown processes $\alpha(t)$, $X(t)$ are approximated by algebraic polynomials:

$$I_{dif}(t) = \frac{du(t)}{dt}\sum_{i=1}^{n1}\tilde{\alpha}_i \cdot t^{i-1} + u(t)\sum_{j=1}^{n2}\tilde{\beta}_j \cdot t^{j-1}. \tag{3}$$

Let us consider the following variable as the residual in the differential form of $I_{dif}(t)$

$$I_{dif}(t) = \tilde{\alpha}(t)\frac{du(t)}{dt} + u(t) - \tilde{X}(t), \tag{4}$$

or taking into account (3):

$$I_{dif}(t) = \frac{du(t)}{dt}\sum_{i=1}^{n1}\tilde{\alpha}_i \cdot t^{i-1} + u(t) - \sum_{j=1}^{n2}\tilde{\beta}_j \cdot t^{j-1}. \tag{5}$$

Instead of the notation of the unknown variables $\tilde{\alpha}_i$, $\tilde{\beta}_j$, let us introduce a single notation for the unknown variables: $\tilde{\alpha}_i = X_i$, $\tilde{\beta}_j = X_{n1+j}$. Through replacing the summation index $n1 + j = i$ in the second sum, we shall obtain the following expression for the residual value in differential form:

$$I_{dif}(t) = \frac{du(t)}{dt}\sum_{i=1}^{n1} t^{i-1}X_i + u(t) - \sum_{i=n1+1}^{n1+n2} t^{i-n1-1}X_i. \tag{6}$$

Through integrating both parts of the relationships (4)–(6) in the limit from t_1 to t, we will obtain expressions for the residual in the integral form $I_{\text{int}}(t)$:

$$I_{\text{int}}(t) = u(t)\tilde{\alpha}(t) - u(t_1)\tilde{\alpha}(t_1) - \int_{t_1}^{t} u(\tau)\frac{d\tilde{\alpha}(\tau)}{d\tau}d\tau +$$

$$+\int_{t_1}^{t} u(\tau)\,d\tau - \int_{t_1}^{t}\tilde{X}(\tau)\,d\tau, \tag{4a}$$

$$I_{\text{int}}(t) = \sum_{i=1}^{n1}\left[t^{i-1}u(t) - t_1^{i-1}u(t_1) - (i-1)\int_{t_1}^{t}\tau^{i-2}u(\tau)d\tau\right]\tilde{\alpha}_i -$$

$$\sum_{j=1}^{n2}\left[\frac{1}{j}\left(t^j - t_1^j\right)\right]\tilde{\beta}_j + \int_{t_1}^{t} u(\tau)\,d\tau, \tag{5a}$$

$$I_{\text{int}}(t) = \sum_{i=1}^{n1}\left[t^{i-1}u(t) - t_1^{i-1}u(t_1) - (i-1)\int_{t_1}^{t}\tau^{i-2}u(\tau)d\tau\right]X_i -$$

$$\sum_{i=n1+1}^{n1+n2}\left[\frac{1}{i-n_1}\left(t^{i-n1} - t_1^{i-n1}\right)\right]X_i + \int_{t_1}^{t} u(\tau)\,d\tau. \tag{6a}$$

A significant number of methods of the basic SLAE construction are elaborated for implementation of the single-channel invariance principle by the set mathematical models which describe the measurement system's dynamic properties. Here is provided the description of several methods for constructing the specified basic SLAE.

The first method represents the integration method on partial intervals

This method is based on the application of residuals in integral form. Having required the residual to be equal to zero in the integral form (6a) written for each of equal partial intervals $[t_1, t_2]$, $[t_2, t_3]$, … , $[t_{n1+n2}, t_{n1+n2+1}]$, i.e. for $[t_k, t_{k+1}]$, $k = 1, 2, \ldots, (n1 + n2)$ partial intervalism, a basic SLAE will be immediately obtained. It is written as

$$\sum_{i=1}^{n1}\left[t_{k+1}^{i-1}u(t_{k+1}) - t_k^{i-1}u(t_k) - (i-1)\int_{t_k}^{t_{k+1}} x^{i-2}u(x)dx\right]X_i -$$

$$\sum_{i=n1+1}^{n1+n2}\left[\frac{1}{i-n1}\left(t_{k+1}^{i-n1} - t_k^{i-n1}\right)\right]X_i = -\int_{t_k}^{t_{k+1}} u(x)\,dx$$

$$k = 1, 2, \ldots, (n1 + n2)$$

Since $t_k = t_1 + T(k-1)$, $t_{k+1} = t_1 + T{\cdot}k$, $T = T0/(n1 + n2)$, where $T0$ is the length of the processes recovery interval $\alpha(t)$, $X(t)$, the SLAE will be recorded as

$$\sum_{i=1}^{n1}\Big\{(t_1+Tk)^{i-1}\cdot(t_1+Tk)-[t_1+T(k-1)]^{i-1}\cdot u[t_1+T(k-1)]-$$

$$(i-1)\int_{t_1+T(k-1)}^{t_1+Tk} x^{i-2}u(x)dx\Big\}X_i-\sum_{i=n1+1}^{n1+n2}\Big\{\frac{1}{i-n1}\Big[(t_1+Tk)^{i-n1}-$$

$$[t_1+T(k-1)]^{i-n1}\Big]\Big\}X_i=-\int_{t_1+T(k-1)}^{t_1+Tk}u(x)dx$$

$$k = 1, 2, \ldots, (n1 + n2).$$

Finally, through introducing the notation

$$I(k,i)=(i-1)\int_{t_1+T(k-1)}^{t_1+Tk} x^{i-2}\cdot u(x)dx,$$

Let us record the basic system in standard form

$$\sum_{i=1}^{n1+n2}A_{ki}X_i=B_k,\quad k=1,2,\ldots,(n1+n2),\qquad \text{ãäå}$$

$$A_{ki}=\begin{cases}(t_1+Tk)^{i-1}\cdot u(t_1+Tk)-[t_1+T(k-1)]^{i-1}\cdot u[t_1+T(k-1)]-I(k,i), & i\le n1\\ -\dfrac{1}{i-n1}\Big[(t_1+Tk)^{i-n1}-[t_1+T(k-1)]^{i-n1}\Big], & i>n1\end{cases} \qquad (7)$$

$$B_k = -I(k, 2).$$

Obviously, the SLAE in the form (7) can also be produced based on the residual in the differential form. For this purpose, it is sufficient to require the equality of integrals of the residual in a differential form to zero, These integrals should be taken within the limits above, specified by the partial intervals $[t_k, t_{k+1}]$, $k = 1, 2, \ldots, (n1 + n2)$.

The second method is similar to the collocation method

This method represents application of the collocation method concept, widely used in numerical analysis when determining solutions to differential equations.

If this method's application is based on a residual value in the differential form, then it is necessary to require that the expression for $I_{dif}(t)$ is equal to zero at the points $t_k = t_1 + T(k-1)$, $k = 1, 2, \ldots, (n1 + n2)$. This requirement immediately produces the sought-for SLAE, which is written as:

$$\left[\frac{du(t)}{dt}\right]\Bigg|_{t=t_k}\sum_{i=1}^{n1}t_k^{i-1}X_i-\sum_{i=n1+1}^{n1+n2}t^{i-n1-1}X_i=-u(t_k)$$

$$k = 1, 2, \ldots, (n1 + n2),\ \ t_k = t_1 + T(k-1),$$

or in standard form

$$\sum_{i=1}^{n1+n2}A_{ki}X_i=B_k,\quad k=1,2,\ldots,(n1+n2),\qquad where$$

$$A_{ki}=\begin{cases}\dfrac{du(t)}{dt}\Big|_{t=t_{1+T(k-1)}}[t_1+T(k-1)]^{i-1}, & i\le n1\\ -[t_1+T(k-1)]^{i-n1-1}, & i>n1\end{cases} \qquad (8)$$

$$B_k = -u([t_1 + T(k-1)].$$

Note that when simulating the process of the measured signal recovery through SLAE (8), the recovery error is by 2 - 3 orders of magnitude greater compared to the error that occurs when using SLAE (7).

The third method is similar to the method of moments

We set forth the method that is based on using the moment method concept, which is also widely applied in numerical analysis when solving differential equations.

We can suppose there is some system of functions $\{\varphi_k(t)\}$, $k = 1, 2, \ldots, (n1 + n2)$ which is complete at $[t_1, t_1 + T0]$. For obtaining the basic SLAE, we require that integrals of the residuals taken with weights of $\varphi_k(t)$, $k = 1, 2, \ldots, (n1 + n2)$ be equal to zero on $[t_1, t_1 + T0]$ interval.

So, if the residual in the differential form is used, then the basic SLAE is determined by the following conditions:

$$\int_{t_1}^{t+T0} I_{dif}(t)\cdot\varphi_k(t)\,dt = 0, \quad k = 1, 2, \ldots, (n1+n2).$$

As the simplest system of functions $\{\varphi_k(t)\}$ we can take the following

$$\varphi_k(t) = t^{k-1}, \quad k = 1, 2, \ldots, (n1 + n2).$$

Through substituting expressions for $I_{dif}(t)$, $\varphi_k(t)$ in the above mentioned conditions, we obtain the following:

$$\sum_{i=1}^{n1}\Big[u(t_1+T0)(t_1+T0)^{i+k-2} - u(t_1)t_1^{\,i+k-2} - (i+k-2)I(k,i)\Big]X_i -$$

$$\sum_{i=n1+1}^{n1+n2}\frac{1}{i+k-n1-1}\Big[(t_1+T0)^{i+k-n1-1} - t_1^{i+k-n1-1}\Big]X_i =$$

$$= -I(k,2),\ I(k,i) = \int_{t_1}^{t_1+T0} u(t)t^{i+k-3}dt.$$

Let's record the system in standard form:

$$\sum_{i=1}^{n1+n2} A_{ki}X_i = B_k, \quad k = 1, 2, \ldots, (n1+n2),$$

$$A_{ki} = \begin{cases} u(t_1+T0)(t_1+T0)^{i+k-2} - u(t_1)(t_1)^{i+k-2} - (i+k-2)\,I(k,i), & i \le n1 \\ -\dfrac{1}{i+k-n1-1}\left[(t_1+T0)^{i+k-n1-1} - t_1^{i+k-n1-1}\right] & i > n1 \end{cases} \tag{9}$$

$$B_k = -I(k, 2).$$

The fourth method represents the reintegration method

We set forth the method for obtaining the basic SLAE, which is based on the repeated integration of a given differential equation which describes the measurement system’s behavior. The concept of the given differential equation’s reintegration was used earlier by V. Streitz, in solving the parametric identification problem [15]. Simultaneously, aiming at reducing errors, Streitz proposed integrating the given differential equation no more than twice, and obtaining the missing SLAEs’ equations through the effect of various known input influences on the system. If in solving the system identification problems, the possibility of influencing the system through various known input signals is easily realizable, then the aforementioned possibility is absent from real dynamic measurements, since the measurement’s basic purpose consists of determination of the unknown input signals. In connection with what is stated above, implementation of the concept of reintegration for obtaining the basic SLAE, other than the one proposed by Streitz which enables us to analytically reduce all repeated integrations to a single integration, is described below.

If $I(t)$ is a residual, then the basic SLAE is produced from the conditions:

$$\underbrace{\int_{t_1}^{t}dt\int_{t_1}^{t}dt\ldots\int_{t_1}^{t}}_{k} I(t)dt = 0, \quad k = 1, 2, \ldots, (n1+n2),$$

where k is the number of repeated integrations.

Let us use the ratio which is known in the analysis:

$$\underbrace{\int_{t_1}^{t} dt \int_{t_1}^{t} dt ... \int_{t_1}^{t}}_{k} I(t)dt = \frac{1}{(k-1)!}\int_{t_1}^{t} (t-z)^{k-1} \cdot I(z)\, dz.$$

Then, the above stated conditions will be expressed through single-time integrals:

$$\int_{t_1}^{t} (t-z)^{k-1} \cdot I(z)\, dz = 0, \quad k = 1, 2, ..., (n1+n2).$$

If the residual in the differential form is taken as the residual, we will obtain

$$\sum_{i=1}^{n1}\left[\int_{t_1}^{t} (t-z)^{k-1} \frac{du(z)}{dz} z^{i-1} dz\right] X_i - \sum_{i=n1+1}^{n1+n2}\left[\int_{t_1}^{t} (t-z)^{k-1} z^{i-n1-1} dz\right] X_i =$$

$$= -\int_{t_1}^{t} (t-z)^{k-1} \cdot u(z)\, dz, \quad k = 1, 2, ..., (n1+n2).$$

Finally, assuming that $t = t_1 + T0$, we shall record this system in standard form:

$$\sum_{i=1}^{n1+n2} A_{ki} X_i = B_k, \quad k = 1, 2, ..., (n1+n2),$$

$$A_{ki} = \begin{cases} \int_{t_1}^{t_1+T0} (t_1 + T0 - z)^{k-1} \frac{du(z)}{dz} z^{i-1} dz, & i \le n1 \\ -\int_{t_1}^{t_1+T0} (t_1 + T0 - z)^{k-1} \cdot z^{i-n_1-1} dz, & i > n1 \end{cases} \tag{10}$$

$$B_k = -\int_{t_1}^{t_1+T0} (t_1 + T0 - z)^{k-1} u(z)\, dz.$$

3.2. DIRECT AND INDIRECT ACCURACY CRITERIA FOR SIGNAL RECOVERY

The stage of selecting direct and indirect criteria which enable us to select the only acceptable solution to the problem from among the abundance of possibilities is also an important stage in the invariance principle's implementation.

Direct criteria

Direct criteria include those that enable us to compare the found estimates of the measured signal with this signal's values based on a particular metric. In computer simulations, such a comparison is possible due to the fact that the measured signal is known, so the role of direct criteria is predominant in assessing the effectiveness of the investigated method of the measured signal recovery

The criterion of the average integral deviation of the estimates $\tilde{\alpha}(t)$, $\tilde{X}(t)$ from the estimated variables $\alpha(t)$, $X(t)$ can be used as the first direct criterion:

$$\rho\alpha = \frac{1}{T0}\int_{t_1}^{t_1+T0} \left|\alpha(t) - \tilde{\alpha}(t)\right| dt, \quad \rho\alpha_{\text{от}} = \frac{\rho\alpha}{\frac{1}{T0}\int_{t_1}^{t_1+T0} \left|\alpha(t)\right| dt}$$
$$\rho X = \frac{1}{T0}\int_{t_1}^{t_1+T0} \left|X(t) - \tilde{X}(t)\right| dt, \quad \rho X_{\text{от}} = \frac{\rho X}{\frac{1}{T0}\int_{t_1}^{t_1+T0} \left|X(t)\right| dt} \tag{11}$$

The maximum deviation criterion can be used as the second direct criterion:

$$\delta\alpha = \max_{t \in [t_1,\, t_1 + T0]} \left|\alpha(t) - \tilde{\alpha}(t)\right|, \qquad \delta\alpha_{\text{от}} = \frac{\delta\alpha}{\frac{1}{T0}\int_{t_1}^{t_1+T0} |\alpha(t)|\, dt}$$

$$\delta X = \max_{t \in [t_1,\, t_1 + T0]} \left|X(t) - \tilde{X}(t)\right|, \qquad \delta X_{\text{от}} = \frac{\delta X}{\frac{1}{T0}\int_{t_1}^{t_1+T0} |X(t)|\, dt} \tag{12}$$

In the future, the standard deviation criterion shall also be used:

$$\sigma\alpha = \left\{\frac{1}{T0}\int_{t_1}^{t_1+T0} [\alpha(t) - \tilde{\alpha}(t)]^2\, dt\right\}^{\frac{1}{2}}, \qquad \sigma\alpha_{\text{от}} = \frac{\sigma\alpha}{\left[\frac{1}{T0}\int_{t_1}^{t_1+T0} \alpha^2(t)\, dt\right]^{\frac{1}{2}}}$$

$$\sigma X = \left\{\frac{1}{T0}\int_{t_1}^{t_1+T0} [X(t) - \tilde{X}(t)]^2\, dt\right\}^{\frac{1}{2}}, \qquad \sigma X_{\text{от}} = \frac{\sigma X}{\left[\frac{1}{T0}\int_{t_1}^{t_1+T0} X^2(t)\, dt\right]^{\frac{1}{2}}} \tag{13}$$

Note that in the above stated three direct criteria, the concept of “the estimate deviation from the estimated variable” is based on the meaning of the difference between these variables. However, there are many other situations where this notion is based on a broader meaning. For example, often in applications the specified notion is based on the meaning that it covers both the difference of estimation and the estimated variable, and a difference of the first derivatives of these variables.

Indirect criteria

As is noted in clause 2.2, indirect criteria can be used for researching the quality of the measured signal recovery, along with direct ones, which take place here below in the simulation. The composite functions that firstly, in its meaning indirectly characterize the proximity degree of $\tilde{X}(t)$, $\tilde{\alpha}(t)$ to $X(t)$, $\alpha(t)$ respectively, and secondly, contain in their structure only the obtained estimates $\tilde{X}(t)$, $\tilde{\alpha}(t)$, other variables known in the process of measurement, but does not contain $X(t)$, $\alpha(t)$ variables, can be referred to indirect criteria for assessing the proximity of the found estimates $\tilde{X}(t)$, $\tilde{\alpha}(t)$ to the sought-for $X(t)$, $\alpha(t)$ variables themselves.

The notion of residuals, which for example, look like the previously given residuals in the differential and integral forms (4), (4A), but may have a different, specially composed structure, can be used as a basis for constructing indirect criteria.

Suppose $I(t)$ is the general notation of the residual. Then, similar to direct criteria, the following simple indirect criteria can be introduced for consideration:

$$\rho I = \frac{1}{T0}\int_{t_1}^{t_1+T0} |I(t)|\, dt, \qquad \rho I_{\text{от}} = \frac{\rho I}{\frac{1}{T0}\int_{t_1}^{t_1+T0} |u(t)|\, dt} \tag{11a}$$

$$\delta I = \max_{t \in [t_1,\, t_1 + T0]} |I(t)|, \qquad \delta I_{\text{от}} = \frac{\delta I}{\frac{1}{T0}\int_{t_1}^{t_1+T0} |u(t)|\, dt} \tag{12a}$$

$$\sigma I = \left[\frac{1}{T0}\int_{t_1}^{t_1+T0} I^2(t)\, dt\right]^{\frac{1}{2}}, \qquad \sigma I_{\text{от}} = \frac{\sigma I}{\left[\frac{1}{T0}\int_{t_1}^{t_1+T0} u^2(t)\, dt\right]^{\frac{1}{2}}} \tag{13a}$$

The $I_u(t)$ residual of the following structure can also be used in the expressions of indirect criteria (11a)–(13a), along with the previously introduced types of residuals in the differential and integral forms (4), (4A):

$$I_u(t) = u(t) - U(t),$$

where $u(t)$ is actual meter reading, and $U(t)$ is the estimate for the indications of the meter reading. The readings that would be taken provided that, firstly, the estimates $\tilde{X}(t)$, $\tilde{\alpha}(t)$ would be taken instead of the unknown variables $X(t)$, $\alpha(t)$, and secondly, $u(t)$, $U(t)$ readings would mutually coincide at the moment of time $t = t1$ (start of the invariance algorithm implementation), will be meant as the reading estimate (the assumed readings). Herewith, we should bear in mind that comparison of $u(t)$, $U(t)$ processes is executed for $t > t1$, and the time $t1$ is close enough to the time of the measuring's start $t = 0$. In other words, if for example at zero initial condition for actual readings (first-order system) we obtain the following:

$$u(t) = \int_0^t e^{-\int_\tau^t \frac{1}{\alpha(y)} dy} \cdot \frac{X(\tau)}{\alpha(\tau)} d\tau, \quad u(t_1) = \int_0^{t_1} e^{-\int_\tau^{t_1} \frac{1}{\alpha(y)} dy} \cdot \frac{X(\tau)}{\alpha(\tau)} d\tau,$$

then the estimate of the device's readings (assumed readings of the device) shall be determined by the following expression:

$$U(t) = u(t_1) \cdot e^{-\int_{t_1}^t \frac{1}{\tilde{\alpha}(y)} dy} + \int_{t_1}^t e^{-\int_\tau^t \frac{1}{\tilde{\alpha}(y)} dy} \cdot \frac{\tilde{X}(\tau)}{\tilde{\alpha}(\tau)} d\tau, \quad t \geq t_1. \tag{14}$$

A sense of the indirect criterion introduction based on the $I_u(t)$ residual, consists of the assumption that if the comparison of true $u(t)$ and assumed $U(t)$ output signals on the $[t_1, t_1 + T0]$ interval reveals that they are, in one sense or another mutually close, then this indirectly indicates the proximity of the sought-for variables $X(t)$, $\alpha(t)$ to the found estimates $\tilde{X}(t)$, $\tilde{\alpha}(t)$ respectively.

For first-order systems, application of the $I_u(t)$ residual is very convenient, since this type of $U(t)$ function is well known, and the time required for its calculation is very brief.

During simulations, the indirect criteria values are not used in the calculation of specific numerical values of errors in the $X(t)$, $\alpha(t)$ process recovery. However, simulation of recovery algorithms for $X(t)$, $\alpha(t)$ processes enables us to record such ranges of variation in indirect criteria variables, or to establish such rules for selecting the degrees of the polynomials which describe $\tilde{X}(t)$, $\tilde{\alpha}(t)$ estimates, at which errors of $X(t)$, $\alpha(t)$ process recovery do not violate acceptable limits.

Thus, despite the fact that values of the indirect criteria variables are not directly engaged in the process of obtaining numerical estimates $\tilde{X}(t)$, $\tilde{\alpha}(t)$, for recoverable $X(t)$, α (t) processes, the need and value of the indirect criteria introduction is that these criteria enable us to select the right direction for searching for these estimates.

3.3. SIMULATION OF MEASURED SIGNAL RECOVERY PROCESSES

Below are given the outcomes of modeling the $X(t)$ recovery processes and $\alpha(t)$ parameter using the single-channel invariance principle for the first-order measurement system. Under this simulation, the model in the form of equation (2) can be taken as the initial mathematical model of this system. In this case, there is no need to introduce generalized unknown variables. Therefore, the simulation process is simple. The simulation process for this case shall be referred to the first layout of the invariance algorithm implementation. If the model in the form of equation (1) is selected as the measurement system's initial mathematical model in the simulation process, then the introduction of generalized unknown variables becomes inevitable, and the simulation process becomes significantly more complicated. The simulation process for this case shall be referred to the second layout of the invariance algorithm implementation.

3.3.1. THE FIRST LAYOUT OF THE INVARIANCE ALGORITHM IMPLEMENTATION

In this layout, the initial data are given by algebraic polynomials:

$$\alpha(t) = \sum_{i=1}^{N1} \alpha_i t^{i-1}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_{N1} \end{pmatrix}, \qquad X(t) = \sum_{j=1}^{N2} \beta_j t^{j-1}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_{N2} \end{pmatrix},$$

$N1$, $N2$ variables, as well as α_i, β_j are given during simulation. However, these variables are unknown for the recovery algorithm for $X(t)$, α (t) processes, and the invariance algorithm uses only the structure of equation (2) which describes the MS's dynamic properties and the $u(t)$ readings of this system.

First of all, the recovery algorithm for the sought-for $X(t)$, $\alpha(t)$ functions is selected through the basic SLAE obtained by one of the methods provided in clause 3.1. When implementing the selected algorithm $\tilde{X}(t)$, $\tilde{\alpha}(t)$ estimates for $X(t)$, $\alpha(t)$ processes are determined as follows:

$$\tilde{\alpha}(t) = \sum_{i=1}^{n1} \tilde{\alpha}_i t^{i-1}, \quad \tilde{\alpha} = \begin{pmatrix} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \\ \overline{\tilde{\alpha}_{n1}} \end{pmatrix}, \qquad \tilde{X}(t) = \sum_{j=1}^{n2} \tilde{\beta}_j t^{j-1}, \quad \tilde{\beta} = \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \overline{\tilde{\beta}_{n2}} \end{pmatrix}, \tag{16}$$

The simulation process consists of setting specific values for $N1$, $N2$, α, β variables, and solving the corresponding basic SLAE, i.e. in determining $\tilde{\alpha}$, $\tilde{\beta}$ estimates for different values of $n1$, $n2$ variables.

Determination of the pair of $n1$, $n2$ variables and the corresponding numerical values of $\tilde{\alpha}$, $\tilde{\beta}$ variables which should be taken as estimates for the sought-for $N1$, $N2$, α, β variables should represent the simulation outcome.

After obtaining a set of $\tilde{X}(t)$, $\tilde{\alpha}(t)$ estimates corresponding to the different values of $n1$, $n2$ variables, it is required to calculate the variables of the corresponding indirect and direct criteria. Analysis of the obtained data consists of: comparison of various values of indirect criteria variables and different pairs of values of $n1$, $n2$ variables corresponding to them; selecting from all pairs ($n1$, $n2$) of a single pair, let us denote it ($\bar{n}^*1$, $\bar{n}^*2$), for which the errors of $X(t)$, $\alpha(t)$ processes recovery are minimal in comparison with the errors corresponding to any other pairs of values of $n1$, $n2$ variables.

This paragraph will state the outcomes of the simulation of the measured signal recovery process obtained by means of each of the methods of the basic SLAE construction submitted in clause 3.1.

The first method of the basic SLAE construction

Let us start with the presentation of the outcomes of simulating the measured signal recovery process related to the case of using the first method of the basic SLAE construction, i.e., using SLAE (7).

As direct criteria for estimating the accuracy of $X(t)$, $\alpha(t)$ processes recovery, the criteria of the average integral, maximum and standard deviations, which are expressed by the ratios (11)–(13) respectively are used. As indirect criteria, the following criteria are used.

Indirect criteria for the average integral deviation of residuals:

$$\rho I_{\text{int}} = \rho I_i = \frac{1}{T0} \int_{t_1}^{t_1+T0} \left| I_{\text{int}}(t) \right| dt, \qquad \rho I_{i\,\text{от}} = \frac{\rho I_i}{\frac{1}{T0} \int_{t_1}^{t_1+T0} |u(t)|\, dt}, \tag{17}$$

where $I_{\text{int}}(t) = \tilde{\alpha}(t)\, u(t) - \tilde{\alpha}(t_1)\, u(t_1) - \int_{t_1}^{t} u(t) \frac{d\tilde{\alpha}(t)}{dt} dt - \int_{t_1}^{t} \tilde{X}(t)\, dt + \int_{t_1}^{t} u(t)\, dt.$

$$\rho I_u(t) = \rho I_u = \frac{1}{T0} \int_{t_1}^{t_1+T0} \left| u(t) - U(t) \right| dt, \qquad \rho I_{u\,\text{от}} = \frac{\rho I_u}{\frac{1}{T0} \int_{t_1}^{t_1+T0} |u(t)|\, dt}, \tag{18}$$

where $U(t) = u(t_1) \cdot e^{-\int_{t_1}^{t} \frac{1}{\tilde{\alpha}(y)} dy} + \int_{t_1}^{t} e^{-\int_{\tau}^{t} \frac{1}{\tilde{\alpha}(y)} dy} \cdot \frac{\tilde{X}(\tau)}{\tilde{\alpha}(\tau)} d\tau, \quad t \geq t_1.$

Indirect criteria for maximum deviation of residuals:

$$\delta I_i = \max \left| I_{\text{int}}(t) \right|, \quad t \in [t_1, t_1 + T0] \qquad \delta I_{i\,\text{от}} = \frac{\delta I_i}{\frac{1}{T0} \int_{t_1}^{t_1+T0} |u(t)|\, dt}, \tag{19}$$

$$\delta I_u = \max \left| u(t) - U(t) \right|, \quad t \in [t_1, t_1 + T0] \qquad \delta I_{u\,\text{от}} = \frac{\delta I_u}{\frac{1}{T0} \int_{t_1}^{t_1+T0} |u(t)|\, dt}, \tag{20}$$

Indirect criterion for the mean-square deviation of the residual:

$$\sigma I_u(t) = \sigma I_u = \left[\frac{1}{T0} \int_{t_1}^{t_1+T0} I_u^2(t)\, dt \right]^{\frac{1}{2}}, \qquad \sigma I_{u\,\text{от}} = \frac{\sigma I_u}{\left[\frac{1}{T0} \int_{t_1}^{t_1+T0} u^2(t)\, dt \right]^{\frac{1}{2}}}, \tag{21}$$

$I_u(t) = u(t) - U(t)$.

The submitted indirect criteria were used in the simulation outcomes presented in tables 1 - 3, in which the relative values of indirect and direct criteria are given as a percentage.

Tables 1 - 3 demonstrate the simulation outcomes related to the following three different versions of the source data:

Source data for the simulation outcomes presented in table 1

$N1$	$N2$	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	t_1	$T0$
3	4	1	10	100	0	2	20	200	2000	0.04	0.1

Source data for the simulation outcomes presented in table 2

$N1$	$N2$	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	t_1	$T0$
4	3	1	10	100	1000	2	20	200	0	0.04	0.1

Source data for the simulation outcomes presented in table 3

$N1$	$N2$	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	t_1	$T0$
4	4	1	10	100	1000	2	20	200	2000	0.04	0.15

The above mentioned numerical values of α_i, β_i variables in the initial polynomials for $\alpha(t)$, $X(t)$ indicate that extremely complex measurement conditions are considered: a very rapid increase in the measurement system inertia, i.e. very rapid deterioration of the system's dynamic properties, accompanied by an even faster increase in $X(t)$ measured characteristic.

Let us proceed to the direct analysis of the simulation outcomes presented in tables 1 - 3 with highlighting the three basic stages of this analysis.

1. Of all the outcomes obtained, such outcomes that do not present physical sense for the measurement systems should be identified and left out of consideration. The mentioned outcomes include such ones that give negative and alternating variables for estimation of $\tilde{\alpha}(t)$. Tables 1 - 3 submit six, three, and six such outcomes, respectively, and they are marked with " – " and "+/–" signs in columns No. 12 of the tables.

Table 1. **Signal recovery *X(t)* and parameter *α(t)*. First method**

$n1$	$n2$	Indirect criteria					Direct criteria						*α(t)* Sign
		$\rho I_{i\,\text{от}}$	$\rho I_{u\,\text{от}}$	$\delta I_{i\,\text{от}}$	$\delta I_{u\,\text{от}}$	$\sigma I_{u\,\text{от}}$	$\rho\alpha_{\text{от}}$	$\rho X_{\text{от}}$	$\delta\alpha_{\text{от}}$	$\delta X_{\text{от}}$	$\sigma\alpha_{\text{от}}$	$\sigma X_{\text{от}}$	
		1	2	3	4	5	6	7	8	9	10	11	12
1	1												–
2	1	0,421	0,044	0,798	0,652	0,047	229,62	205,67	352,06	262,33	231,722	193,49	+
1	2												–
2	2												+/–
3	2												–
2	3	$4,693\cdot10^{-6}$	$1,156\cdot10^{-4}$	$1,771\cdot10^{-5}$	$3,172\cdot10^{-4}$	$1,458\cdot10^{-4}$	98,66	96,15	141,70	159,80	98,802	96,38	+
3	3												+/–
4	3												–
3	4	$5,679\cdot10^{-14}$	$1,987\cdot10^{-14}$	$1,406\cdot10^{-13}$	$7,806\cdot10^{-14}$	$2,582\cdot10^{-14}$	$6,185\cdot10^{-6}$	$6,026\cdot10^{-6}$	$9,09\cdot10^{-6}$	$1,041\cdot10^{-5}$	$6,19\cdot10^{-6}$	$6,036\cdot10^{-6}$	+
4	4	$4,547\cdot10^{-14}$	$1,616\cdot10^{-14}$	$1,295\cdot10^{-13}$	$1,295\cdot10^{-13}$	$2,184\cdot10^{-14}$	$2,463\cdot10^{-5}$	$2,398\cdot10^{-5}$	$3,645\cdot10^{-5}$	$4,19\cdot10^{-5}$	$2,464\cdot10^{-5}$	$2,402\cdot10^{-5}$	+
5	4	$7,202\cdot10^{-14}$	$2,406\cdot10^{-14}$	$2,005\cdot10^{-13}$	$2,005\cdot10^{-13}$	$3,762\cdot10^{-14}$	$1,617\cdot10^{-3}$	$1,575\cdot10^{-5}$	$2,405\cdot10^{-3}$	$2,775\cdot10^{-3}$	$1,618\cdot10^{-3}$	$1,576\cdot10^{-3}$	+
4	5	$7,036\cdot10^{-14}$	$2,680\cdot10^{-14}$	$3,187\cdot10^{-13}$	$1,916\cdot10^{-13}$	$5,151\cdot10^{-14}$	$5,09\cdot10^{-3}$	$5,025\cdot10^{-3}$	$8,577\cdot10^{-3}$	$9,894\cdot10^{-3}$	$5,253\cdot10^{-3}$	$5,224\cdot10^{-3}$	+
5	5	$9,014\cdot10^{-14}$	$4,670\cdot10^{-14}$	$5,125\cdot10^{-13}$	$3,335\cdot10^{-13}$	$8,995\cdot10^{-14}$	0,045	0,044	0,074	0,086	0,046	0,046	+
6	5	$6,345\cdot10^{-14}$	$2,505\cdot10^{-14}$	$2,779\cdot10^{-13}$	$2,768\cdot10^{-13}$	$4,328\cdot10^{-14}$	0,045	0,045	0,073	0,085	0,046	0,046	+
5	6	$6,655\cdot10^{-14}$	$3,206\cdot10^{-14}$	$3,009\cdot10^{-13}$	$1,774\cdot10^{-13}$	$4,868\cdot10^{-14}$	0,323	0,325	0,659	0,764	0,354	0,361	+
6	6	$7,117\cdot10^{-14}$	$3,560\cdot10^{-14}$	$4,311\cdot10^{-13}$	$4,311\cdot10^{-13}$	$6,346\cdot10^{-14}$	23,58	23,65	47,03	54,64	25,53	25,93	+

Table 2. **Signal recovery *X(t)* and parameter *α(t)*. First method**

$n1$	$n2$	Indirect criteria					Direct criteria						*α(t)* Sign
		$\rho I_{i\,\text{от}}$	$\rho I_{u\,\text{от}}$	$\delta I_{i\,\text{от}}$	$\delta I_{u\,\text{от}}$	$\sigma I_{u\,\text{от}}$	$\rho\alpha_{\text{от}}$	$\rho X_{\text{от}}$	$\delta\alpha_{\text{от}}$	$\delta X_{\text{от}}$	$\sigma\alpha_{\text{от}}$	$\sigma X_{\text{от}}$	
		1	2	3	4	5	6	7	8	9	10	11	12
1	1	0,19	0,089	$3,789\cdot10^{-2}$	0,159	0,099	94,25	91,21	127,5	115,53	95,133	91,928	+
2	1	$3,173\cdot10^{-3}$	0,025	$6,902\cdot10^{-3}$	0,044	0,027	96,82	93,81	147,47	128,22	97,432	94,31	+
1	2	0,025	0,026	0,052	0,054	0,029	74,96	71,40	125,22	108,85	79,24	75,011	+
2	2												+/–
3	2	$1,741\cdot10^{-5}$	$1\cdot10^{-4}$	$7,045\cdot10^{-5}$	$4,444\cdot10^{-4}$	$1,653\cdot10^{-4}$	94,64	91,76	158,19	133,11	95,089	92,12	+
2	3												+/–
3	3	$1,334\cdot10^{-6}$	$3,189\cdot10^{-5}$	$6,529\cdot10^{-6}$	$1,229\cdot10^{-4}$	$4,863\cdot10^{-5}$	98,77	95,87	167,6	139,88	98,922	96,008	+
4	3	$6,547\cdot10^{-14}$	$1,446\cdot10^{-14}$	$1,644\cdot10^{-13}$	$1,644\cdot10^{-13}$	$1,943\cdot10^{-14}$	$3,668\cdot10^{-8}$	$3,559\cdot10^{-8}$	$6\cdot10^{-8}$	$5\cdot10^{-8}$	$3,679\cdot10^{-8}$	$3,568\cdot10^{-8}$	+
3	4												–
4	4	$5,968\cdot10^{-14}$	$2,094\cdot10^{-14}$	$2,574\cdot10^{-13}$	$2,574\cdot10^{-13}$	$3,22\cdot10^{-14}$	$6,247\cdot10^{-5}$	$6,068\cdot10^{-5}$	$1,087\cdot10^{-4}$	$8,979\cdot10^{-5}$	$6,248\cdot10^{-5}$	$6,069\cdot10^{-5}$	+
5	4	$7,493\cdot10^{-14}$	$2,024\cdot10^{-14}$	$2,293\cdot10^{-13}$	$2,293\cdot10^{-13}$	$2,777\cdot10^{-14}$	$3,315\cdot10^{-4}$	$3,19\cdot10^{-4}$	$6,309\cdot10^{-4}$	$5,195\cdot10^{-4}$	$3,404\cdot10^{-4}$	$3,259\cdot10^{-4}$	+
4	5	$8,67\cdot10^{-14}$	$3,894\cdot10^{-14}$	$3,402\cdot10^{-13}$	$2,076\cdot10^{-13}$	$6,351\cdot10^{-14}$	$1,528\cdot10^{-3}$	$1,484\cdot10^{-3}$	$2,68\cdot10^{-3}$	$2,206\cdot10^{-3}$	$1,527\cdot10^{-3}$	$1,484\cdot10^{-3}$	+
5	5	$1,162\cdot10^{-13}$	$5,687\cdot10^{-14}$	$6,371\cdot10^{-13}$	$3,893\cdot10^{-13}$	$1,052\cdot10^{-13}$	$4,78\cdot10^{-3}$	$4,661\cdot10^{-3}$	$8,156\cdot10^{-3}$	$6,697\cdot10^{-3}$	$4,731\cdot10^{-3}$	$4,626\cdot10^{-3}$	+
6	5	$9,473\cdot10^{-14}$	$3,324\cdot10^{-14}$	$4,102\cdot10^{-13}$	$2,509\cdot10^{-13}$	$6,722\cdot10^{-14}$	0,065	0,061	0,149	0,122	0,071	0,066	+
5	6	$8,714\cdot10^{-14}$	$2,874\cdot10^{-14}$	$3,421\cdot10^{-13}$	$2,163\cdot10^{-13}$	$5,767\cdot10^{-14}$	$2,894\cdot10^{-3}$	$2,622\cdot10^{-3}$	$9,255\cdot10^{-3}$	$7,583\cdot10^{-3}$	$3,797\cdot10^{-3}$	$3,405\cdot10^{-3}$	+
6	6	$1,52\cdot10^{-13}$	$1,096\cdot10^{-13}$	$1,328\cdot10^{-12}$	$8,219\cdot10^{-13}$	$2,089\cdot10^{-13}$	1,373	1,299	3,112	2,545	1,491	1,394	+

Table 3. **Signal recovery *X(t)* and parameter *α(t)*. First method**

$n1$	$n2$	Indirect criteria					Direct criteria						*α(t)* Sign
		$\rho I_{i\ от}$	$\rho I_{u\ от}$	$\delta I_{i\ от}$	$\delta I_{u\ от}$	$\sigma I_{u\ от}$	$\rho\alpha_{от}$	$\rho X_{от}$	$\delta\alpha_{от}$	$\delta X_{от}$	$\sigma\alpha_{от}$	$\sigma X_{от}$	
		1	2	3	4	5	6	7	8	9	10	11	12
1	1												–
2	1	$1{,}105 \cdot 10^{-3}$	$1{,}129 \cdot 10^{-2}$	$2{,}832 \cdot 10^{-3}$	$1{,}518 \cdot 10^{-2}$	$1{,}198 \cdot 10^{-2}$	98,521	96,625	168,656	165,894	99,217	97,484	+
1	2												–
2	2												+/–
3	2												+/–
2	3												+/–
3	3												–
4	3	$7{,}462 \cdot 10^{-8}$	$2{,}712 \cdot 10^{-6}$	$4{,}963 \cdot 10^{-7}$	$1{,}141 \cdot 10^{-5}$	$4{,}65 \cdot 10^{-6}$	99,45	97,53	201,28	198,28	99,559	97,814	+
3	4	$7{,}463 \cdot 10^{-8}$	$2{,}728 \cdot 10^{-6}$	$4{,}963 \cdot 10^{-7}$	$1{,}144 \cdot 10^{-5}$	$4{,}676 \cdot 10^{-6}$	99,46	97,54	201,30	198,30	99,568	97,823	+
4	4	$1{,}052 \cdot 10^{-13}$	$9{,}29 \cdot 10^{-15}$	$3{,}775 \cdot 10^{-13}$	$3{,}775 \cdot 10^{-13}$	$1{,}484 \cdot 10^{-14}$	$1{,}568 \cdot 10^{-5}$	$1{,}538 \cdot 10^{-5}$	$3{,}228 \cdot 10^{-5}$	$3{,}181 \cdot 10^{-5}$	$1{,}571 \cdot 10^{-5}$	$1{,}543 \cdot 10^{-5}$	+
5	4	$8{,}517 \cdot 10^{-14}$	$1{,}362 \cdot 10^{-14}$	$4{,}283 \cdot 10^{-13}$	$4{,}283 \cdot 10^{-13}$	$2{,}037 \cdot 10^{-14}$	$2{,}052 \cdot 10^{-4}$	$2{,}012 \cdot 10^{-4}$	$4{,}29 \cdot 10^{-4}$	$4{,}227 \cdot 10^{-4}$	$2{,}057 \cdot 10^{-4}$	$2{,}021 \cdot 10^{-4}$	+
4	5	$1{,}272 \cdot 10^{-13}$	$1{,}531 \cdot 10^{-14}$	$6{,}255 \cdot 10^{-13}$	$6{,}255 \cdot 10^{-13}$	$2{,}213 \cdot 10^{-14}$	$2{,}052 \cdot 10^{-4}$	$2{,}012 \cdot 10^{-4}$	$4{,}29 \cdot 10^{-4}$	$4{,}226 \cdot 10^{-4}$	$2{,}057 \cdot 10^{-4}$	$2{,}021 \cdot 10^{-4}$	+
5	5	$5{,}656 \cdot 10^{-13}$	$5{,}644 \cdot 10^{-13}$	$4{,}832 \cdot 10^{-12}$	$2{,}976 \cdot 10^{-12}$	$8{,}399 \cdot 10^{-13}$	0,039	0,038	0,088	0,086	$3{,}992 \cdot 10^{-2}$	$3{,}923 \cdot 10^{-2}$	+
6	5	$2{,}431 \cdot 10^{-13}$	$1{,}577 \cdot 10^{-13}$	$1{,}398 \cdot 10^{-12}$	$9{,}675 \cdot 10^{-13}$	$2{,}341 \cdot 10^{-13}$	0,0135	0,0133	0,023	0,0225	$1{,}265 \cdot 10^{-2}$	$1{,}242 \cdot 10^{-2}$	+
5	6	$2{,}676 \cdot 10^{-13}$	$1{,}557 \cdot 10^{-13}$	$1{,}398 \cdot 10^{-12}$	$8{,}628 \cdot 10^{-13}$	$2{,}331 \cdot 10^{-13}$	0,0135	0,0133	0,023	0,0226	$1{,}266 \cdot 10^{-2}$	$1{,}242 \cdot 10^{-2}$	+
6	6	$6{,}268 \cdot 10^{-13}$	$7{,}109 \cdot 10^{-13}$	$6{,}928 \cdot 10^{-12}$	$4{,}456 \cdot 10^{-12}$	$1{,}159 \cdot 10^{-12}$	9,35	9,18	26,99	26,60	10,71	10,53	+

2. After excluding the outcomes without physical sense from consideration, all the simulation outcomes presented in each of tables 1-3 can be divided into two groups on a provisional basis, which differ sharply (incomparably) from each other in terms of the values of the selected indirect criteria. In other words, during the sequential increase in $n1$, $n2$ variables and during the transition from one pair ($n1$, $n2$) to another one the pairs appear (denote them (n^*1, n^*2)), the indirect criteria values for which show a sharp decrease. All pairs (n^*1, n^*2), which are characterized by a sharp decrease in the values of the selected indirect criteria, shall be referred to the basic group of pairs of the $n1$, $n2$ variable values. Variable values of the indicated sharp decrease in indirect criteria for tables 1-3 make from ~ 10^5 to 10^{10} times, while the values of the indirect criteria for all pairs within the basic group (n^*1, n^*2) are very close to each other. The Basic groups of pairs for tables 1-3 are as follows:

(3, 4) ≤ (n^*1, n^*2) ≤ (6, 6) – for table 1;
(4, 3) ≤ (n^*1, n^*2) ≤ (6, 6) – for table 2;
(4, 4) ≤ (n^*1, n^*2) ≤ (6, 6) – for table 3.

It is obvious that the sets (n^*1, n^*2) for these three cases can be narrowed through assuming the pair (5, 5) as the upper limit.

3. It remains to select from the variety of possible solutions - the set of pairs (n^*1, n^*2) - a single one ($\bar{n}^*1$, $\bar{n}^*2$) for which $\bar{n}^*1$, $\bar{n}^*$ can serve as estimates for the sought-for $N1$, $N2$, and found $\tilde{X}(t)$, $\tilde{\alpha}(t)$ variables, corresponding to the pair ($\bar{n}^*1$, $\bar{n}^*2$), can serve as estimates for the sought-for $X(t)$, $\alpha(t)$. It is quite natural to select from the set of pairs ($n*1$, $n*2$) within the basic group the pair ($\bar{n}^*1$, $\bar{n}^*2$) for which $\bar{n}*1$ is the smallest of all $n*1$ numbers, and the value $\bar{n}^*2$ is the smallest of all $n*2$ numbers. As follows from tables 1-3, the pointed out pairs are, respectively, as stated below:

$$(\bar{n}^*1 = 3, \bar{n}^*2 = 4),\ (\bar{n}^*1 = 4, \bar{n}^*2 = 3),\ (\bar{n}^*1 = 4, \bar{n}^*2 = 4).$$

Finally, we verify that that the found pairs ($\bar{n}^*1$, $\bar{n}^*2$) exactly coincide with the source pairs ($N1$, $N2$) for each one out of tables 1-3, and the variables of the corresponding errors in the $X(t)$, $\alpha(t)$ process recovery, estimated based on the direct criteria variables, prove to be minimal in comparison with the errors related to other pairs (n^*1, n^*2) within the basic group.

From all this follows the conclusion: for solving the measured signal recovery problem precisely during the measurement process, it is necessary to stipulate the use of special logging sub-routines at $t \in [t_1, t_1 + T0]$ of all data related to $u(t)$ readings of the measurement system, which are used while solving the basic SLAE for each of the proposed search options.

Thus, the essence of the general conclusion is that for $X(t)$ input signal and $\alpha(t)$ parameter of the measurement system, which represent algebraic polynomials, the invariance algorithm enables us, firstly to accurately determine the sought-for degrees of these polynomials, and secondly, to determine $X(t)$, $\alpha(t)$ processes through their estimates $\tilde{X}(t)$, $\tilde{\alpha}(t)$ with minimal errors.

As follows from tables 1-3, when using the invariance algorithm and the specified rule for the single solution selection, the errors in determining α_i, β_i parameters of $X(t)$, $\alpha(t)$ processes, and the errors in determining these processes themselves are incomparably lower than can be required practically, even under the most rigid requirements for the dynamic measurement accuracy, i.e., the specified errors are almost negligibly small.

In conclusion, we note that here above, when analyzing the patterns of indirect criteria behavior at changing $n1$, $n2$ variables, we did not focus on any specific indirect criteria, but formulated these patterns relative to all the indirect criteria used simultaneously. This is associated with the fact that the patterns of all these indirect criteria behavior at changing $n1$, $n2$ variables are identical. However, note, that sensitivity of indirect criteria that use $I_u(t)$ and $I_{\text{int}}(t)$ residuals is higher than, for example, the sensitivity of indirect criteria that use $I_{dif}(t)$ residual, and this is obviously important at selecting a single solution from among a set of possible solutions.

The second method of the basic SLAE construction

Naturally, it is required to clarify how the above-described patterns of $X(t)$, $\alpha(t)$ process recovery based on their $\tilde{X}(t)$, $\tilde{\alpha}(t)$ estimates are characteristic of other methods of the basic SLAE construction. Table 4 provides the simulation outcomes related to the basic SLAE obtained through the second of the methods described in clause 3.1. This SLAE is written as (8).

In this case $\rho\alpha_{\text{от}}$, $\rho X_{\text{от}}$ criteria written in the previous form were used as direct criteria. As indirect criteria were used $\rho I_u(t) = \rho I_u$ criteria written in the previous form, and $\rho I_{\text{dif}} = \rho I_d$, written as:

$$\rho I_{dif}(t) = \rho I_d = \frac{1}{T0}\int_{t_1}^{t_1+T0} \left| I_{dif}(t) \right| dt, \qquad \rho I_{d\,\text{от}} = \frac{\rho I_d}{\frac{1}{T0}\int_{t_1}^{t_1+T0} \left| u(t) \right| dt}, \tag{22}$$

where $I_{dif}(t) = \tilde{\alpha}(t)\dfrac{du(t)}{dt} + u(t) - \tilde{X}(t)$.

As follows from the data submitted in table 4, in this case the same patterns of changes in the indirect criteria variables when changing $n1$, $n2$ are found, which were described above relative to the first method of obtaining the basic SLAE. Therefore, the rule of finding a pair ($\bar{n}^*1$, $\bar{n}^*2$) representing the sought-for outcome remains of course, unchanged.

Source data for the simulation outcomes presented in table 4, are written as:

$N1$	$N2$	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	t_1	$T0$
4	3	1	10	100	1000	2	20	200	0	0.04	0.1

Table 4. **Recovery of $X(t)$ signal and $\alpha(t)$ parameter. Second method**

$n1$	$n2$	Indirect criteria				Direct criteria		$\tilde{\alpha}(t)$ Sign
		ρI_d	$\rho I_{d\,от}$	ρI_u	$\rho I_{u\,от}$	$\rho\alpha_{от}$	$\rho X_{от}$	
1	1	$5.4\cdot10^{-3}$	3.366	$1.222\cdot10^{-3}$	0.761	93.972	91.031	+
2	1	$9.379\cdot10^{-4}$	0.585	$1.918\cdot10^{-4}$	0.12	96.901	93.896	+
1	2	0.013	7.874	$2.833\cdot10^{-4}$	0.177	57.28	53.275	+
2	2							+/–
3	2	$7.041\cdot10^{-6}$	$4.389\cdot10^{-3}$	$1.728\cdot10^{-6}$	$1.077\cdot10^{-3}$	94.776	91.89	+
2	3							+/–
3	3	$6.03\cdot10^{-7}$	$3.759\cdot10^{-4}$	$2.771\cdot10^{-7}$	$1.728\cdot10^{-4}$	98.837	95.942	+
4	3	$1.095\cdot10^{-13}$	$6.826\cdot10^{-11}$	$1.421\cdot10^{-15}$	$8.858\cdot10^{-13}$	$1.307\cdot10^{-5}$	$1.269\cdot10^{-5}$	+
3	4							–
4	4	$1.346\cdot10^{-13}$	$8.39\cdot10^{-11}$	$1.551\cdot10^{-15}$	$9.671\cdot10^{-13}$	$6.145\cdot10^{-5}$	$5.968\cdot10^{-5}$	+
5	4	$1.559\cdot10^{-13}$	$9.72\cdot10^{-11}$	$3.197\cdot10^{-15}$	$1.993\cdot10^{-12}$	$3.601\cdot10^{-3}$	$3.466\cdot10^{-3}$	+
4	5	$2.653\cdot10^{-13}$	$1.654\cdot10^{-10}$	$8.175\cdot10^{-15}$	$5.096\cdot10^{-12}$	0.015	0.015	+
5	5	$1.007\cdot10^{-13}$	$6.275\cdot10^{-11}$	$1.712\cdot10^{-15}$	$1.067\cdot10^{-12}$	$3.308\cdot10^{-3}$	$3.21\cdot10^{-3}$	+
6	5	$1.067\cdot10^{-13}$	$6.653\cdot10^{-11}$	$1.924\cdot10^{-16}$	$1.199\cdot10^{-13}$	0.097	0.092	+
5	6	$1.047\cdot10^{-13}$	$6.525\cdot10^{-11}$	$1.381\cdot10^{-16}$	$8.608\cdot10^{-14}$	0.012	0.011	+
6	6	$3.097\cdot10^{-13}$	$1.931\cdot10^{-10}$	$9.98\cdot10^{-15}$	$6.221\cdot10^{-12}$	6.86	6.493	+

Note that now, when the source data is received, errors in the recovery of $X(t)$, $\alpha(t)$ processes are approximately three orders of magnitude greater compared to the errors corresponding to this case in the first method of the basic SLAE construction, and provided in table 2.

However, despite the increase in error of $X(t)$, $\alpha(t)$ process recovery, the specified error for the selected solution ($\bar{n}^*1$, $\bar{n}^*2$), as well as at using the first method of the basic SLAE construction, proves to be negligible.

The third and fourth methods of the basic SLAE construction

The simulation outcomes related to the third method (analog of the moment method) and the fourth method (re-integration method) are submitted in tables 5 and 6. Direct and indirect criteria used in these tables have the same meaning as in tables 1 - 3, and 4.

The simulation outcomes submitted in tables 5 and 6 refer to the source data:

$N1$	$N2$	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	t_1	$T0$
3	4	1	10	100	0	2	20	200	2000	0.04	0.1

Table 5. **Recovery of $X(t)$ signal and $\alpha(t)$ parameter. Third method**

$n1$	$n2$	Indirect criteria		Direct criteria		$\tilde{\alpha}(t)$ Sign
		ρI_u	$\rho I_{u\,от}$	$\rho\alpha_{от}$	$\rho X_{от}$	
1	1					–
2	1	$9.779\cdot10^{-5}$	$5.001\cdot10^{-2}$	39.042	36.360	+
1	2	$7.828\cdot10^{-5}$	$4\cdot10^{-2}$	103.841	100.989	+
2	2					+/–
3	2	$7.77\cdot10^{-3}$	3.973	100.68	98.071	+
2	3	$2.684\cdot10^{-7}$	$1.372\cdot10^{-4}$	98.75	96.24	+
3	3					+/–
4	3	$7.459\cdot10^{-7}$	$3.814\cdot10^{-4}$	100.8	98.2	+
3	4	$1.3\cdot10^{-14}$	$6.514\cdot10^{-12}$	$2.438\cdot10^{-3}$	$2.375\cdot10^{-3}$	+
4	4	$5.2\cdot10^{-14}$	$2.677\cdot10^{-11}$	$1.066\cdot10^{-1}$	$1.039\cdot10^{-1}$	+
5	4	$2.25\cdot10^{-13}$	$1.152\cdot10^{-10}$	10.245	9.979	+
4	5	$1.3\cdot10^{-13}$	$6.654\cdot10^{-11}$	19.536	19.278	+
5	5	$2.28\cdot10^{-13}$	$1.166\cdot10^{-10}$	10.873	10.596	+
6	5					+/–
5	6	$1.879\cdot10^{-12}$	$9.609\cdot10^{-10}$	12.413	11.296	+
6	6	$1.394\cdot10^{-11}$	$7.126\cdot10^{-9}$	100.677	98.041	+

Table 6. **Recovery of $X(t)$ signal and $\alpha(t)$ parameter. Fourth method**

$n1$	$n2$	Indirect criteria		Direct criteria		$\tilde{\alpha}(t)$ Sign
		ρI_u	$\rho I_{u\,от}$	$\rho\alpha_{от}$	$\rho X_{от}$	
1	1	$2.814 \cdot 10^{-4}$	$1.439 \cdot 10^{-1}$	107.58	104.49	+
2	1	$9.779 \cdot 10^{-5}$	$5 \cdot 10^{-2}$	39.04	36.36	+
1	2	$7.828 \cdot 10^{-5}$	$4.003 \cdot 10^{-2}$	103.84	100.99	+
2	2					+/–
3	2	$7.77 \cdot 10^{-3}$	3.973	100.68	98.07	+
2	3	$2.684 \cdot 10^{-7}$	$1.372 \cdot 10^{-4}$	98.75	96.24	+
3	3					+/–
4	3	$7.458 \cdot 10^{-7}$	$3.813 \cdot 10^{-4}$	100.8	98.2	+
3	4	10^{-15}	$4.45 \cdot 10^{-13}$	$2.365 \cdot 10^{-4}$	$2.304 \cdot 10^{-4}$	+
4	4	$1.8 \cdot 10^{-14}$	$9.285 \cdot 10^{-12}$	$6.245 \cdot 10^{-2}$	$6.083 \cdot 10^{-2}$	+
5	4	$1.42 \cdot 10^{-13}$	$7.279 \cdot 10^{-11}$	9.18	8.95	+
4	5	$1.54 \cdot 10^{-13}$	$7.883 \cdot 10^{-11}$	20.35	20.08	+
5	5	$4.63 \cdot 10^{-13}$	$2.366 \cdot 10^{-10}$	136	134	+
6	5	$7.79 \cdot 10^{-13}$	$3.983 \cdot 10^{-10}$	104	101.5	+
5	6	$2.8 \cdot 10^{-13}$	$1.432 \cdot 10^{-10}$	167.7	167.5	+
6	6					+/–

Analysis of the simulation outcomes obtained in this case is conducted similar to the previous cases, and the patterns of $X(t)$, $\alpha(t)$ processes recovery pursuant to their $\tilde{X}(t)$, $\tilde{\alpha}(t)$ estimates remain the same as were described when using the first method of the basic SLAE obtaining. Note only that for $N1 = 3$, $N2 = 4$, errors of the third and fourth methods prove to be three and two orders of magnitude greater, respectively, compared to the error that occurred when using the first method of the basic SLAE construction. However, this fact is not very important, since values of the errors themselves in the recovery of $X(t)$, $\alpha(t)$ processes using the third and fourth methods of the basic SLAE construction can be considered as negligible.

Thus, based on the outcomes of simulating the first layout of the invariance algorithm implementation by means of four methods of the basic SLAE construction which are different in terms of accuracy, the following can be concluded:

1. The single-channel invariance principle is an effective means of eliminating the influence of parametric effects on the dynamic measurement accuracy.

2. A simple rule for selecting a single solution to the problem of recovery of $X(t)$ measured signal and $\alpha(t)$ parameter of the measurement system from a set of possible solutions is elaborated. This rule is general and valid irrespective of the method of the basic SLAE construction, and the type of indirect criteria used.

3. Errors in recovery of the measured $X(t)$ signal and $\alpha(t)$ parameter of the measurement system using the invariance algorithm can be considered negligible for dynamic measurements: the greatest error occurs when using the third method of the basic SLAE construction, and it makes $< 2.5 \cdot 10^{-3}$ %.

Along with the described methods of the basic SLAE construction, the integral least squares method was also used for simulating the process of the measured signal recovery. Herewith, the qualitative conclusions regarding the signal recovery process are quite similar to those stated here above relative to the four methods of the basic SLAE construction described in clause 3.1. As for the signal recovery errors when using the integral least squares method, they are negligible for dynamic measurements as well, although they are slightly higher than in previous cases.

We conclude the presentation regarding the first layout of the invariance algorithm implementation by considering the quality of recovery of the measured $X(t)$ signal and $\alpha(t)$ parameter depending on location of $[t_1, t_1 + T0]$ recovery interval relative to the measurement process start $t = 0$.

Let us turn again to the extreme measurement conditions considered here above: under the selected variables of α_i, β_i parameters, rapid deterioration of the measurement system's dynamic properties accompanied by a rapid increase in the measured signal itself, is observed in the measurement process. The mentioned exceptional conditions are far from the actual measurement conditions. However, such measurement situations present interest since they enable us to study the invariance algorithm in very unfavorable circumstances.

Therefore, we see fit to investigate the change in the quality of the invariance algorithm functioning at successive intervals $[t_1^{(k)}, t_1^{(k)} + T0]$, $k = 1, 2, 3\ldots$ of recovery of fast-flowing $X(t)$, $\alpha(t)$ processes with increasing $t_1^{(k)}$ time point – the recovery process's start.

It is obvious that, due to the exceptional nature of change in $\alpha(t)$ parameter, a certain time point occurs during the measurement process, after which the inertia of the measurement system becomes extremely great, and therefore the system's dynamic sensitivity becomes extremely low. In the specified measurement zone, no algorithms for the sought-for $X(t)$, $\alpha(t)$ processes recovery pursuant to $u(t)$ readings of the measurement system, including of course, the invariance algorithm as well, enable us to achieve the goal.

Thus, it is required to set the last interval $[t_1^{(k)}, t_1^{(k)} + T0]$, at which recovery of $X(t)$, $\alpha(t)$ processes is still possible, and the recovery quality is satisfactory. Herewith, it is meant of course, that the invariance algorithm produces an acceptable outcome for all previous recovery intervals for $X(t)$, $\alpha(t)$ processes.

The simulation outcomes that enable us to estimate the quality of the invariance algorithm's functioning on several consecutive intervals of $X(t)$, $\alpha(t)$ processes' recovery are provided in table 7. These outcomes are obtained by means of the first method of the basic SLAE construction and $\rho I_{u\,\text{от}}$ indirect criterion. Herewith, it is taken into consideration that selection of a single solution from among a set of possible solutions has already been executed, which enabled us to establish that $n1 = N1 = 3$, $n2 = N2 = 4$. In addition to the basic ones, Table 7 also presents some additional values that enable us to trace in detail the evolution of the measurement situation itself, in which $\alpha(t)$ parameter has the meaning of the MS "time constant" which is time variant.

Let us go directly to analysis of the data submitted in Table 7. Recovery of $X(t)$, $\alpha(t)$ processes starts at $t_1 = 0.02$ sec and ends at $t_1 = 0.97$ sec, herewith the following five consecutive recovery intervals are considered: [0.02; 0.12], [0.12; 0.27], [0.27; 0.52], [0.52; 0.97], [0.97; 1.42].

The measurement situations reflected in the first and second rows of Table 7 are more or less close to feasible ones. Note that the situation reflected in the second row can already be considered as the maximum possible one.

The measurement process at the first [0.02; 0.12] interval is accompanied by an approximately three-fold increase of $\alpha(t)$, i.e. by three-fold deterioration of the measurement system's dynamic properties. Herewith, $u(t)$ system readings at the end of the first interval lag behind the respective measured $X(t)$ in ~ 40 times, and the change rate of $u'(t)$ system readings lags behind the change rate of the measured signal $X'(t)$ in ~ 50 times at this point.

Table 7. **Signal recovery *X(t)* and parameter *α(t)* with sharply deteriorating properties of MS (*n1 = N1 = 3, n2 = N2 = 4*).**

t_1	$T0$	Average integral estimates			$\frac{\alpha(t_1)}{\alpha(t_1+T0)}$	$\frac{u(t_1)}{u(t_1+T0)}$	$\frac{u'(t_1)}{u'(t_1+T0)}$	$\frac{X(t_1)}{X(t_1+T0)}$	$\frac{X'(t_1)}{X'(t_1+T0)}$
		$\rho I_{u\,\text{от}}$	$\rho\alpha_{\text{от}}$	$\rho X_{\text{от}}$					
0,02	0,1	$1{,}222\cdot10^{-14}$	$1{,}362\cdot10^{-6}$	$1{,}326\cdot10^{-6}$	1,24	0,04	1,981	2,496	30,4
					3,64	0,271	2,875	10,736	154,4
0,12	0,15	$1{,}246\cdot10^{-14}$	$6{,}622\cdot10^{-5}$	$6{,}508\cdot10^{-5}$	3,64	0,271	2,875	10,736	154,4
					10,99	0,892	5,501	61,346	565,4
0,27	0,25	$1{,}978\cdot10^{-14}$	$3{,}961\cdot10^{-3}$	$3{,}923\cdot10^{-3}$	10,99	0,892	5,501	61,346	565,4
					33,24	2,872	10,374	347,696	$1{,}85\cdot10^{3}$
0,52	0,45	$3{,}089\cdot10^{-14}$	$7{,}3\cdot10^{-2}$	$7{,}2\cdot10^{-2}$	33,24	2,872	10,374	347,696	$1{,}85\cdot10^{3}$
					104,79	9,552	19,328	$2{,}035\cdot10^{3}$	$6{,}053\cdot10^{3}$
0,97	0,45	$9{,}898\cdot10^{-15}$	7,807	7,778	104,79	9,552	19,328	$2{,}035\cdot10^{3}$	$6{,}053\cdot10^{3}$
					216,84	20,271	28,316	$6{,}16\cdot10^{3}$	$1{,}269\cdot10^{3}$

When recovering $X(t)$, $\alpha(t)$ processes at the second [0.12; 0.27] interval, at the end of the specified interval $\alpha(t)$ parameter takes the value that is almost ten times greater compared to the value of this parameter at the start of the measured signal recovery process, i.e. at $t_1 = 0.02$. As a result, at the end of the second interval $u(t)$ readings of the measurement system lag behind from $X(t)$ already in ~ 70 times, and $u'(t)$ rate lags behind $X'(t)$ even in more than 100 times. That is, even at the second interval of $X(t)$, $\alpha(t)$ processes recovery, the technical means used for measurements cannot be considered a measuring tool at all, in the conventional sense.

Despite these peculiarities of the measurement situation, as follows from Table 7, application of the invariance algorithm enables us to recover $X(t)$, $\alpha(t)$ processes on both the first and the second time intervals with a very high degree of accuracy.

As for the recovery intervals using the invariance algorithm for $X(t)$, $\alpha(t)$ processes corresponding to the third and fourth rows of Table 7, despite the sufficiently high degree of recovery accuracy at these intervals, these outcomes are practically irrelevant to the actual possible measurement conditions and the measuring tools used during this procedure.

Thus, bearing in mind the above stated issues, we conclude that even in the considered exceptional measurement situation, the invariance algorithm enables us to restore the sought-for $X(t)$, $\alpha(t)$ processes with a high degree of accuracy up to the fourth interval [0.52; 0.97] inclusive. From the moment of time of $t \geq t_1 = 0.97$ sec, the errors in recovery of $X(t)$, $\alpha(t)$ processes become noticeable however, and as was noted above, the measurement conditions in this time zone do not comply with the ones which are actually possible.

The data submitted in Table 7 also indicate to some extent the manifestation of the effect which is the result of removing the recovery interval $[t_1, t_1 + T0]$ from the start of the measurement process $t = 0$, and of approximating this interval to the “dynamically steady-state stage” of measurement. This effect was pointed out at stating the content of the invariance principle which is single-channel from the physical standpoint in clause 2.2, and consists of the following: this recovery interval transition leads to reducing the legitimacy of the invariance algorithm application and, consequently, increasing the error of the sought-for processes recovery. As follows from the data submitted in Table 7, the error of $X(t)$ signal recovery on the fourth interval proves to be more than four orders of magnitude greater compared to the recovery error on the first interval.

The raised issue will be discussed in more detail at the end of this chapter.

3.3.2. THE SECOND LAYOUT OF THE INVARIANCE ALGORITHM IMPLEMENTATION

The second layout of the invariance algorithm implementation is based on the use of equation (1) as the source model of the measurement system:

$$\frac{du(t)}{dt} + a(t)u(t) = a(t)X(t),$$

which is a special case ($n = 1$) of the second general model (1.II) MS with concentrated parameters.

The second layout is more complicated than the first one, since in this case it is necessary to introduce generalized (intermediate) unknown variables. Consideration of the invariance algorithm implementation in relation to the model in form (1) is important, since the application of known physical laws during the derivation of equations describing the MS's behavior often results in namely equations in form (1), in which the basic physical parameter $a(t)$ is included in a linear way.

The general approach to constructing the invariance algorithm for the second MS's general model with concentrated parameters was considered in clause 2.3. Now, for the purpose of the simulation, let us give concrete expression to the invariance algorithm construction in relation to the first-order measurement systems. In this case, with the purpose of preserving the notation unity in this chapter, we will slightly step aside from the notations used in clause 2.3.

Outcomes of the simulated invariance algorithm implementation stated here below are obtained by means of the first method of the basic SLAE construction and the residual in the integral form.

So, we will suppose the estimates $\tilde{a}(t)$, $\tilde{X}(t)$ of the parameter $\alpha(t)$ and the measured signal $X(t)$ are approximated through algebraic polynomials:

$$\tilde{a}(t) = \sum_{i=1}^{n1} \tilde{\alpha}_i t^{i-1}, \quad \tilde{X}(t) = \sum_{j=1}^{n2} \tilde{\beta}_j t^{j-1}. \tag{23}$$

Note that in this case the meaning of $\tilde{\alpha}_i$ variables is already different compared to the first layout of the invariance algorithm's implementation.

Since the residual in the integral form which corresponds to the model (1) is written as

$$I_{\text{int}}(t) = u(t) - u(t_1) + \int_{t_1}^{t} \tilde{a}(\tau)u(\tau)d\tau - \int_{t_1}^{t} \tilde{a}(\tau)\tilde{X}(\tau)d\tau,$$

then taking into account (23) we obtain:

$$I_{\text{int}}(t) = u(t) - u(t_1) + \sum_{i=1}^{n1}\left[\int_{t_1}^{t} \tau^{i-1}u(\tau)d\tau\right]\tilde{\alpha}_i -$$

$$\sum_{i=1}^{n1}\sum_{j=1}^{n2}\left[\frac{1}{i+j-1}\left(t^{i+j-1} - t_1^{i+j-1}\right)\right]\tilde{\alpha}_i\tilde{\beta}_j. \tag{24}$$

During the construction of the invariance algorithm for the second general model in clause 2.3, it was noted that in this case the invariance algorithm is of nonlinear nature. For converting this algorithm into a linear one, it is required to introduce the generalized $\gamma_1, \gamma_2, \ldots, \gamma_q$ unknown variables for consideration. As follows from the form of the given residual (24), the additive component containing the products of $\tilde{\alpha}_i \cdot \tilde{\beta}_j$.are the sources of nonlinearities.

In order to give visualization to the procedure for introducing generalized (intermediate) unknown variables, let us set forth this procedure for specific degrees $n1$, $n2$ of approximating polynomials.

So, suppose $n1 = 4$, $n2 = 4$, then the residual (24) will be written as follows

$$I_{\text{int}}(t) = u(t) - u(t_1) + \sum_{i=1}^{n1}\left[\int_{t_1}^{t} \tau^{i-1}u(\tau)d\tau\right]\tilde{\alpha}_i - (t - t_1)\,\tilde{\alpha}_1\tilde{\beta}_1 -$$

$$\frac{1}{2}(t^2 - t_1^2)\,(\tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1) - \frac{1}{3}(t^3 - t_1^3)\,(\tilde{\alpha}_1\tilde{\beta}_3 + \tilde{\alpha}_2\tilde{\beta}_2 + \tilde{\alpha}_3\tilde{\beta}_1) -$$

$$\frac{1}{4}(t^4 - t_1^4)\,(\tilde{\alpha}_1\tilde{\beta}_4 + \tilde{\alpha}_2\tilde{\beta}_3 + \tilde{\alpha}_3\tilde{\beta}_2 + \tilde{\alpha}_4\tilde{\beta}_1) - \frac{1}{5}(t^5 - t_1^5)\,(\tilde{\alpha}_2\tilde{\beta}_4 + \tilde{\alpha}_3\tilde{\beta}_3 + \tilde{\alpha}_4\tilde{\beta}_2) - \tag{25}$$

$$\frac{1}{6}(t^6 - t_1^6)\,(\tilde{\alpha}_3\tilde{\beta}_4 + \tilde{\alpha}_4\tilde{\beta}_3) - \frac{1}{7}(t^7 - t_1^7)\,\tilde{\alpha}_4\tilde{\beta}_4.$$

Structure of γ_j generalized unknowns is obvious:

$$\gamma_1 = \tilde{\alpha}_1\tilde{\beta}_1, \quad \gamma_2 = \tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1, \quad \gamma_3 = \tilde{\alpha}_1\tilde{\beta}_3 + \tilde{\alpha}_2\tilde{\beta}_2 + \tilde{\alpha}_3\tilde{\beta}_1,$$

$$\gamma_4 = \tilde{\alpha}_1\tilde{\beta}_4 + \tilde{\alpha}_2\tilde{\beta}_3 + \tilde{\alpha}_3\tilde{\beta}_2 + \tilde{\alpha}_4\tilde{\beta}_1, \quad \gamma_5 = \tilde{\alpha}_2\tilde{\beta}_4 + \tilde{\alpha}_3\tilde{\beta}_3 + \tilde{\alpha}_4\tilde{\beta}_2,$$

$$\gamma_6 = \tilde{\alpha}_3\tilde{\beta}_4 + \tilde{\alpha}_4\tilde{\beta}_3, \quad \gamma_7 = \tilde{\alpha}_4\tilde{\beta}_4.$$

Let us denote by $n3$ the number of new unknown variables γ_j, which is equal to $n3 = n1 + n2 - 1 = 7$. Therefore, the total number of unknown variables that are now to be determined is equal to $n1 + n3 = 2n1 + n2 - 1$, i.e. is greater by ($n1$-1) than the number of original unknown variables $\tilde{\alpha}_i$, $i = 1, 2, \ldots, n1$ и $\tilde{\beta}_j$, $j = 1, 2, \ldots, n2$.

Now the residual will be written as follows

$$I_{\text{int}}(t) = u(t) - u(t_1) + \sum_{i=1}^{n1}\left[\int_{t_1}^{t} \tau^{i-1}u(\tau)d\tau\right]\tilde{\alpha}_i - \sum_{j=1}^{n2}\frac{1}{j}(t^j - t_1^j)\,\gamma_j.$$

The basic SLAE obtained by means of the first method will be written as follows

$$\sum_{i=1}^{n1}\left[\int_{t_k}^{t_{k+1}} \tau^{i-1}u(\tau)d\tau\right]\tilde{\alpha}_i - \sum_{j=1}^{n3}\frac{1}{j}(t_{k+1}^j - t_k^j)\,\gamma_j = u(t_k) - u(t_{k+1}), \tag{26}$$

$$k = 1, \ldots, (n1 + n3) = 11,\ t_k = t_1 + T(k-1),\ t_{k+1} = t_1 + Tk,\ T = \frac{T0}{k} = \frac{T0}{11}.$$

Through entering a single notation of unknown variables

$$\alpha_i = X_i,\ i = 1, 2, \ldots, n1,\ \gamma_j = X_{j+n1},\ j = 1, 2, \ldots, n3,$$

we obtain

$$\sum_{i=1}^{n1}\left[\int_{t_k}^{t_{k+1}}\tau^{i-1}u(\tau)d\tau\right]X_i-\sum_{j=1}^{n3}\frac{1}{j}(t_{k+1}^j-t_k^j)\,X_{j+n1}=u(t_k)-u(t_{k+1}).$$

Through replacing $j + n1 = i$ indices in the second sum, we obtain

$$\sum_{i=1}^{n1}\left[\int_{t_k}^{t_{k+1}}\tau^{i-1}u(\tau)d\tau\right]X_i-\sum_{i=n1+1}^{n1+n3}\left[\frac{1}{i-n1}(t_{k+1}^{i-n1}-t_k^{i-n1})\right]X_i=u(t_k)-u(t_{k+1}). \tag{27}$$

$$k = 1, 2, \ldots, (n1 + n3) = 11.$$

Let's record the system in standard form:

$$\sum_{i=1}^{n1+n3}A_{ki}X_i=B_k,\quad k = 1, 2, \ldots, (n1 + n3) = 11 \tag{28}$$

$$A_{ki}=\begin{cases}\displaystyle\int_{t_k}^{t_{k+1}}\tau^{i-1}u(\tau)d\tau, & i\le n1\\ \displaystyle -\frac{1}{i-n1}(t_{k+1}^{i-n1}-t_k^{i-n1}), & i>n1\end{cases}$$

$$B_k = u(t_k) - u(t_{k+1}).$$

After solving the system (28) we find the source unknown constants $\tilde\alpha_i$, $i = 1, 2, \ldots, n1$ и $\tilde\beta_j$, $j = 1, 2, \ldots, n2$:

$$\tilde\alpha_i = X_i,\ i = 1, 2, \ldots, n1,\quad \tilde\beta_1=\frac{\gamma_1}{\tilde\alpha_1}=\frac{X_5}{X_1},\quad \tilde\beta_2=\frac{X_6-X_2\tilde\beta_1}{X_1},$$

$$\tilde\beta_3=\frac{X_7-X_2\tilde\beta_2-X_3\tilde\beta_1}{X_1},\quad \tilde\beta_4=\frac{X_8-X_2\tilde\beta_3-X_3\tilde\beta_2-X_4\tilde\beta_1}{X_1}.$$

The found intermediate unknown variables γ_5, γ_6, γ_7, т. е. X_9, X_{10}, X_{11} were not used, since this was not required. Note however, that without the introduction of these “extra” generalized unknowns, it would be impossible to linearize the invariance algorithm.

At changing $n1$, $n2$ values type of the system (28) remains the same. However, the number of source unknown variables $\tilde\alpha_i$, $\tilde\beta_j$, the number of generalized unknown variables γ_i, as well as the structure of the relationship between the source and intermediate unknown variables all change.

The equations which establish the relationship between the source and intermediate unknown variables for different combinations of $n1$, $n2$ values are submitted here below. The conclusions of all these equations shall be omitted, since they are obtained similarly to the case ($n1 = 4$, $n2 = 4$) illustrated above:

1. $n1 = 1$, $n2 = 1$, $n3 = 1$, $i = 1, 2$, $k = 1, 2$, $T = T0/2$

$$\tilde\alpha_1 = X_1;\quad \tilde\beta_1 = X_2/X_1$$

2. $n1 = 2$, $n2 = 1$, $n3 = 2$, $i = 1, \ldots, 4$, $k = 1, \ldots, 4$, $T = T0/4$

$$\tilde\alpha_1 = X_1;\quad \tilde\alpha_2 = X_2;\quad \tilde\beta_1 = X_3/X_1$$

3. $n1 = 1$, $n2 = 2$, $n3 = 2$, $i = 1, \ldots, 3$, $k = 1, \ldots, 3$, $T = T0/3$

$$\tilde\alpha_1 = X_1;\quad \tilde\beta_1 = X_2/X_1;\quad \tilde\beta_2 = X_3/X_1$$

4. $n1 = 2$, $n2 = 2$, $n3 = 3$, $i = 1, \ldots, 5$, $k = 1, \ldots, 5$, $T = T0/5$

$$\tilde\alpha_1 = X_1;\quad \tilde\alpha_2 = X_2;\quad \tilde\beta_1 = X_3/X_1;\quad \tilde\beta_2 = (X_4 - X_2\cdot\tilde\beta_1)/X_1$$

5. $n1 = 3$, $n2 = 1$, $n3 = 3$, $i = 1, \ldots, 6$; $k = 1, \ldots, 6$, $T = T0/6$

$$\tilde\alpha_i = X_i;\quad i = 1, \ldots, 3,\quad \tilde\beta_1 = X_4/X_1$$

6. $n1 = 3$, $n2 = 2$, $n3 = 4$, $i = 1, \ldots, 7$; $k = 1, \ldots, 7$, $T = T0/7$

$$\tilde\alpha_i = X_i;\quad i = 1, \ldots, 3;\quad \tilde\beta_1 = X_4/X_1;\quad \tilde\beta_2 = (X_5 - X_2\cdot\tilde\beta_1)/X_1$$

7. $n1 = 2$, $n2 = 3$, $n3 = 4$, $i = 1, \ldots, 6$, $k = 1, \ldots, 6$, $T = T0/6$

$$\tilde{\alpha}_1 = X_1;\quad \tilde{\alpha}_2 = X_2;\quad \tilde{\beta}_1 = X_3/X_1;\quad \tilde{\beta}_2 = (X_4 - X_2\cdot\tilde{\beta}_1)/X_1;$$
$$\tilde{\beta}_3 = (X_5 - X_2\cdot\tilde{\beta}_2)/X_1$$

8. $n1 = 3,\ n2 = 3,\ n3 = 5,\ i = 1, \ldots, 8,\ k = 1, \ldots, 8,\ T = T0/8$

$$\tilde{\alpha}_i = X_i;\quad i = 1, \ldots, 3;\quad \tilde{\beta}_1 = X_4/X_1;\quad \tilde{\beta}_2 = (X_5 - X_2\cdot\tilde{\beta}_1)/X_1;$$
$$\tilde{\beta}_3 = (X_6 - X_2\cdot\tilde{\beta}_2 - X_3\cdot\tilde{\beta}_1)/X_1$$

9. $n1 = 4,\ n2 = 1,\ n3 = 4,\ i = 1, \ldots, 8,\ k = 1, \ldots, 8,\ T = T0/8$

$$\tilde{\alpha}_i = X_i;\quad i = 1, \ldots, 4;\quad \tilde{\beta}_1 = X_5/X_1$$

10. $n1 = 4,\ n2 = 2,\ n3 = 5,\ i = 1, \ldots, 9,\ k = 1, \ldots, 9,\ T = T0/9$

$$\tilde{\alpha}_i = X_i;\quad i = 1, \ldots, 4;\quad \tilde{\beta}_1 = X_5/X_1;\quad \tilde{\beta}_2 = (X_6 - X_2\cdot\tilde{\beta}_1)/X_1$$

11. $n1 = 4,\ n2 = 3,\ n3 = 6,\ i = 1, \ldots, 10,\ k = 1, \ldots, 10,\ T = T0/10$

$$\tilde{\alpha}_i = X_i;\quad i = 1, \ldots, 4;\quad \tilde{\beta}_1 = X_5/X_1;\quad \tilde{\beta}_2 = (X_6 - X_2\cdot\tilde{\beta}_1)/X_1;$$
$$\tilde{\beta}_3 = (X_7 - X_2\cdot\tilde{\beta}_2 - X_3\cdot\tilde{\beta}_1)/X_1$$

12. $n1 = 3,\ n2 = 4,\ n3 = 6,\ i = 1, \ldots, 9,\ k = 1, \ldots, 9,\ T = T0/9$

$$\tilde{\alpha}_i = X_i;\quad i = 1, \ldots, 3;\quad \tilde{\beta}_1 = X_4/X_1;\quad \tilde{\beta}_2 = (X_5 - X_2\cdot\tilde{\beta}_1)/X_1;$$
$$\tilde{\beta}_3 = (X_6 - X_2\cdot\tilde{\beta}_2 - X_3\cdot\tilde{\beta}_1)/X_1;\quad \tilde{\beta}_4 = (X_7 - X_2\cdot\tilde{\beta}_3 - X_3\cdot\tilde{\beta}_2)/X_1$$

Table 8 presents the outcomes of the simulation for the recovery of the $X(t)$ measured signal process, and $a(t)$ parameter using the second layout of the invariance algorithm's implementation. These outcomes enable us to investigate the quality of these properties recovery, as well as to estimate the difference between the outcomes of $X(t)$, $a(t)$ processes recovery obtained at using the first and second layouts of the invariance algorithm implementation.

Algebraic polynomials are taken as models of the source $X(t)$, $a(t)$ processes:

$$a(t) = \sum_{i=1}^{N1} \alpha_i t^{i-1}, \quad X(t) = \sum_{j=1}^{N2} \beta_j t^{j-1}$$

Table 8. **Recovery of the $X(t)$ signal and the $\alpha(t)$ parameter under the second layout of implementation for the invariance algorithm**
($t_1 = 0,04$; $T0 = 0,1$)

$N1, N2, n1, n2$	Indirect criteria				Direct criteria	
	ρI_i	$\rho I_{i\,\text{от}}$	ρI_u	$\rho I_{u\,\text{от}}$	$\rho a_{\text{от}}$	$\rho X_{\text{от}}$
$N1 = 3,\ N2 = 4,$ $n1 = 3,\ n2 = 4$	$3.192\cdot10^{-7}$	$4.029\cdot10^{-5}$	$3.007\cdot10^{-7}$	$3.795\cdot10^{-5}$	$8.188\cdot10^{-5}$	$1.269\cdot10^{-4}$
$N1 = 4,\ N2 = 3,$ $n1 = 4,\ n2 = 3$	$4.551\cdot10^{-8}$	$5.894\cdot10^{-6}$	$4.098\cdot10^{-8}$	$5.307\cdot10^{-6}$	$6.4\cdot10^{-5}$	$6.406\cdot10^{-5}$
$N1 = 4,\ N2 = 4,$ $n1 = 4,\ n2 = 4$	$3.184\cdot10^{-7}$	$3.309\cdot10^{-5}$	$2.944\cdot10^{-7}$	$3.059\cdot10^{-5}$	$7.859\cdot10^{-4}$	$7.412\cdot10^{-4}$

and three options of values of pair ($N1$, $N2$) are considered:

$$N1 = 3,\ N2 = 4,\ N1 = 4,\ N2 = 3,\ N1 = 4,\ N2 = 4.$$

The numerical values of α_i, β_j variables in the polynomials are taken from the matrix-columns:

$$\alpha = \begin{pmatrix} 1 \\ 10 \\ 100 \\ 1000 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 2 \\ 20 \\ 200 \\ 2000 \end{pmatrix}$$

Since the procedure of selection from among simulation outcomes within the basic group (n^*1, n^*2), and then the only pair ($\bar{n}^*1$, $\bar{n}^*2$, i.e. selection of the single solution for the set problem from among a variety of possible solutions, in the second layout of the invariance algorithm implementation is similar to that used in the algorithm implementation under the first layout. This step is not submitted in the table, but instead this selection's outcomes are presented, namely, the options of the invariance algorithm implementation for ($\bar{n}^*1 = 3$, $\bar{n}^*2 = 4$), ($\bar{n}^*1 = 4$, $\bar{n}^*2 = 3$), ($\bar{n}^*1 = 4$, $\bar{n}^*2 = 4$).

As follows from the outcomes submitted in table 8, the recovery of $X(t)$, $a(t)$ processes based on their $\tilde{X}(t)$, $\tilde{a}(t)$ estimates in the second layout of the invariance algorithm implementation takes place with a high accuracy degree.

Further, comparison of the outcomes submitted in Table 8 with those submitted in Tables 1-3 indicate that the transition from the first to the second layout of the invariance algorithm implementation led to an increase in the error of the sought-for $X(t)$, $a(t)$ process recovery by several orders of magnitude in the studied cases. The effect of increasing errors during transition from the first to the second layout of the invariance algorithm implementation is mainly explained by the fact that the orders of the basic SLAEs in the second implementation layout are higher ($n1 - 1$) compared to the orders of the corresponding basic SLAEs in the first implementation layout. However, the specified increase in errors is not, generally speaking, of fundamental value, since the errors of the sought-for $X(t)$, $a(t)$ functions recovery are negligible in both cases.

3.4. SIMULATION OF THE PROCESSES OF STANDARD DETERMINED SIGNAL RECOVERY

In the practice of measuring non-electric variables, the following types of measured signals are conventionally considered as standard signals:

- the signal which represents the time-independent quantity;
- the signal which represents the linearly time-varying function;
- the signal that varies in time pursuant to the harmonic law;
- the signal which represents the stationary random time-varying function.

The first three types of signals are classified as deterministic signals, and they will be studied in this paragraph.

The role of these standard signals in the measurement theory and practice is very significant. This is stipulated by the fact that they are often found in the dynamic measurement practice, and the fact that the quality of measurement devices as stationary measurement systems is often estimated based on their response exactly to these types of signals. In this regard, in the field of non-electric quantities measurement, special test facilities and installations are elaborated that generate the above specified signals with various degrees of approximation, in order to execute the experimental study of the dynamic properties of measuring devices as stationary measurement systems. However, the mentioned installations and test facilities are almost useless in terms of researching the dynamic properties of measuring devices, which are considered as non-stationary dynamic systems.

At the point of transition to researching non-stationary measurement systems, a natural question arises concerning the types of the MS parameter change laws which can be considered as typical. If the field of non-electric quantity measurement as a whole is meant, then currently there are no outcomes of any general reliable systematic studies which enable us to answer the set question unequivocally. Therefore, the theoretical studies conducted so far which were aimed at estimating the impact of time variability of MS parameters on dynamic measurement accuracy, were based only on assumptions regarding the possible laws of MS parameter change. The specified assumptions were based on analysis of the peculiarities of the processes in the objects whose properties are subject to measurement.

Analysis of the specifics of the functioning of a large number of engineering and processing facilities, the parameters of which are to be measured, suggests the conclusion that while measuring deterministic signals, the following two types of patterns of deterministic time-variable parameters of measuring devices are most often seen. Firstly, the MS's parameters vary monotonically between a source, and final values, and this monotonous change can be either of an increasing or of a decreasing nature. Secondly, changes of MS parameters in time can be regular, which, under certain conditions and with some assumptions, can be approximated by the harmonic law in the simplest case.

In the context of measuring a signal that represents a random process, in particular, a stationary random process, the nature of the MS's parameter change under such conditions can also be described by a random time-varying function; in particular, the stationary random function. A typical and very important example of the latter case is the measurement of the temperature of liquid and gas turbulence flows - i.e. flows, which are characterized by a random variation in time, both of the temperature and flow rate. In this case, the time-variance of the factor of convective heat exchange between thermal detector and flow, acts as the source of $a(t)$ parameter variability in the measuring device, which in turn, is a consequence of the dependence of the convective heat exchange factor on the liquid and gas flow rates.

This paragraph considers the invariance algorithm implementation for the three above-mentioned deterministic standard signals with the time-variable MS parameter. This selection is also explained by the fact that on recovering the specified standard signals through the invariance algorithm, the basic peculiarities of this algorithm's operation are clearly manifested.

The invariance algorithm implementation for the case when the measured $X(t)$ signal and $a(t)$ MS parameter represent the stationary random processes will be discussed in chapter 5.

Concerning the distribution of partial intervals

Before proceeding to the presentation of the outcomes of the invariance algorithm implementation for standard deterministic signals, let us focus on one particular issue regarding the selection of time reference points for the MS readings.

If the measurement cycle duration makes τ, then during this time it is very important to fix such volume of the required source information, mainly associated with $u(t)$ readings of the measurement system, which will be sufficient for different options of the invariance algorithm's implementation - for different combinations of $n1$, $n2$, $T0$ source variables, as well as different positions of $[t_1, t_1 + T0]$ interval of the sought-for $X(t)$, $a(t)$ process recovery on the time axis. If the raised issue is not essential for the simulation, then this issue, although it is of a technical nature, is important for the practical invariance algorithm implementation under actual measurement conditions. The situation gets more complicated if the invariance algorithm implementation must be executed directly during the measurement procedure. In such cases, application of a special sub-routine for recording, storing and using the measurement information is required for the appropriate arrangement of the measurement process, which ultimately, converts the measurement system into a measuring and computing system.

Generally speaking, the experiment planning in the specified sense can be extremely diverse, which is definitely determined by the specific statement of the measurement task, the object's nature and the measurement conditions.

Without touching this issue thoroughly, we present one particular rule for the distribution of partial intervals on the time axis, which is quite simple in its execution under the set $n1$, $n2$ variables.

Let's assume that the first method of the basic SLAE construction is used, and the first interval on which the invariance algorithm is implemented is $[t_1, t_1 + T01]$, where $T01$ is the length of the first recovery interval. The number of partial intervals k which compose the first interval, equal to the number of unknown variables in the basic SLAE, and length $T1$ of the partial interval on this interval (discrete pitch size on the first interval) are determined by the following expressions:

$$k = n1 + n3, \quad n3 = n1 + n2 - 1, \quad T1 = T01/k.$$

The reference points t_i of the output signal values $u(t_i)$ shall be as follows

$$t_i = t_1 + T1(i - 1), \quad i = 1, 2, \ldots, (k + 1).$$

If, for example, $n1 = 4$, $n2 = 4$, i.e. $n3 = 7$, and the first interval length is $T01$, then:

$$k = 11, \quad T1 = T01/11, \quad t_1 = t_1, \quad t_2 = t_1 + T1, \ldots, t_{11} = t_1 + 10T1,$$

$$t_{12} = t_1 + 11T1.$$

Thus, at the first interval of the invariance algorithm implementation, the number of partial intervals k, the value of the discrete pitch size $T1$, which are necessary in the future, are calculated based on the source data $n1$, $n2$, $T01$, and the reference points t_i of $u(t_i)$ output signal are determined. Passing to the second and subsequent intervals, we assume that the exponents of approximating polynomials for $\tilde{X}(t)$, $\tilde{a}_i(t)$ remain the same as on the first interval, i.e. the number of unknown variables in SLAE, and therefore the number of partial intervals k on the second and subsequent intervals remains the same as in the first interval.

The second interval, on which the invariance algorithm will be implemented, is selected in such a way that firstly, it would include the full first interval, i.e. $t = t_1$ point is the start of the second interval as well as of the first one, and secondly, in contrast to the first interval, the discrete pitch size $T2$ is used on this interval among the source data for the second interval. In this case, length of the second interval $T02$ is determined already by the source data $n1$, $n2$, $T2$, namely, $T02 = T2 \cdot k = T2(n1 + n3)$. Finally, and this is very important, discrete pitch size $T2$ on the second interval should be taken equal to the length of the full first interval, i.e. $T2 = T01$.

So, for the second interval we have the following source data: $n1$, $n2$, $T01$, based on which length of the second interval and the reference point are determined:

$$T02 = T2 \cdot k = T01(n1 + n3), \quad t_i = t_1 + T2(i - 1) = t_1 + T01(i - 1),$$

$$i = 1, \ldots, (k + 1)$$

Thus, from among the values of the output signal registered on the first interval for the invariance algorithm implementation on it, only two values $u(t_1)$, $u(t_1 + T01)$, i.e. the values of the output signal at the boundaries of the first interval $[t_1, t_1 + T01]$, will be required when implementing the invariance algorithm on the second interval. However in this case - as the output signal values at the boundaries of the first partial interval $[t_1, t_1 + T2]$ of the second interval $[t_1, t_1 + T02]$. Therefore, when implementing the invariance algorithm on the second interval, it will be required to record the values of the output signal $u(t_i)$ only in points:

$$t_i = t_1 + T2\,(i - 1) = t_1 + T01(i - 1), \quad i = 3, \ldots, (k + 1).$$

Distribution of partial intervals for subsequent intervals of the invariance algorithm implementation is constructed in a similar way. For example, the third interval's start, as well as the start of all intervals, is determined by $t = t_1$ point. Length of the partial interval $T3$ on the third interval is determined by the length of the full second interval $T02$, i.e. $T3 = T02$, and length of the full third interval makes $T03 = T3 \cdot k = T02 \cdot k$. Since the entire second interval $[t_1, t_1 + T02]$ will now act as the first partial interval $[t_1, t_1 + T3]$ of the third interval $[t_1, t_1 + T03]$, then recording the values of $u(t)$ output signal when implementing the invariance algorithm on the third interval is required only at points

$$t_i = t_1 + T3(i - 1) = t_1 + T02(i - 1), \quad i = 3, \ldots, (k + 1).$$

Obviously, in the stated rule for partial intervals distribution length of each interval, starting from the second one, is k times greater compared to the previous interval length.

Peculiarity of this rule is that, while passing from the first to the subsequent intervals of the sought-for processes recovery, we increasingly inevitably include the segments of the steady-state measurement stage in $[t_1, t_1 + T0i]$ recovery interval, and this obviously provokes an increase in the error of the measured signal recovery. Therefore, while using the described rule for partial interval distribution in the invariance algorithm implementation procedure, it is important to set that last interval of the sought-for $X(t)$, $a(t)$ processes recovery, on which the invariance algorithm application remains valid.

Recovery of the first standard signal

Let us focus on the simulation outcomes related to the first standard signal, assuming that the measured $X(t)$ signal represents the time-independent quantity $X(t) = \beta_1 = \text{const}$, and that $a(t)$ parameter varies exponentially, increasing in the measurement process from its source $a_н$ value to the final $a_к$ value:

$$X(t) = \beta_1, \quad N2 = 1; \quad a(t) = a_к - (a_к - a_н)e^{-\mu t}, \quad \mu = \text{const}.$$

The corresponding outcomes are submitted in table 9, where p is the ordinal number of the interval on which the sought-for $X(t)$, $a(t)$ processes are recovered; TP is the partial interval (discrete pitch size) for the p interval; $T0P$ is the length of the full p interval.

Table 9. **Recovery of the first standard signal**

P	$t_1 \le t \le t_1 + T0P$	TP	$T0P$	Indirect criteria		Direct criteria		cond 1(A) / cond 2(A)
				$\rho I_{i\,от}$	$\rho I_{u\,от}$	$\rho a_{от}$	$\rho X_{от}$	
1	$n1 = 3$. $0.04 \le t \le 0.14$	0.017	0.1	$1.06 \cdot 10^{-3}$	$1.03 \cdot 10^{-3}$	$5.737 \cdot 10^{-2}$	$5.467 \cdot 10^{-2}$	2.5 •1012 / 1.6 •1012
2	$n1 = 3$, $0.04 \le t \le 0.64$	0.1	0.6	$6.639 \cdot 10^{-3}$	$5.689 \cdot 10^{-3}$	$6.9 \cdot 10^{-2}$	$5.749 \cdot 10^{-2}$	1.14 •109 / 8.73 •108
3	$n1 = 3$. $0{,}04 \le t \le 3.64$	0.6	3.6	$1.689 \cdot 10^{-2}$	$7.752 \cdot 10^{-3}$	$6.244 \cdot 10^{-2}$	$1.765 \cdot 10^{-2}$	4.67 •106 / 3.35 •106
4	$n1 = 4$. $0.04 \le t \le 0.64$	0.075	0.6	$5.454 \cdot 10^{-4}$	$4.667 \cdot 10^{-4}$	$3.215 \cdot 10^{-3}$	$2.992 \cdot 10^{-3}$	1.26 •1013 / 8.7 •1012
5	$n1 = 4$. $0.04 \le t \le 4.84$	0.6	4.8	$2.63 \cdot 10^{-3}$	$9.7 \cdot 10^{-4}$	$8.738 \cdot 10^{-3}$	$1.714 \cdot 10^{-3}$	1.42 •109 / 1.08 •109

As the values of the constants included in the expression for $a(t)$, $X(t)$ are taken as the following: $\beta_1 = 100$; $a_н = 1$, $a_к = 2$, $\mu = 0.04$. $\tilde{a}_i(t)$, $\tilde{X}(t)$ estimates of the sought-for $a(t)$, $X(t)$ are approximated by algebraic polynomials:

$$\tilde{a}(t) = \sum_{i=1}^{n1} \tilde{\alpha}_i t^{i-1}, \quad \tilde{X}(t) = \sum_{j=1}^{n2} \tilde{\beta}_j t^{j-1}.$$

Note that in this case, the time variation law for the sought-for $a(t)$ parameter does not coincide with the approximation in time law for $\tilde{a}_i(t)$ estimate: the exponential variation in time of the sought-for parameter is compared to the variation in its estimate pursuant to the law described by the polynomial in $n1$ degree.

As previously, selection of the only solution for the set problem from a variety of possible solutions is executed through selecting from the outcomes of the invariance algorithm implementation within the basic group of pairs (n^*1, n^*2), followed by selection of the only pair ($\bar{n}^*1$, $\bar{n}^*2$) from the basic group.

When simulating the process of the sought-for $X(t)$, $a(t)$ recovery for the first standard signal, we've slightly simplified the specified stage of the single solution selection, by having selected immediately $\bar{n}^*1 = 3$, we changed only $n2$ to determine $\bar{n}^*2$. This enabled us to establish almost immediately that $\bar{n}^*2 = N2$, i.e. $\bar{n}^*2 = 1$. Therefore, Table 9 does not submit the outcomes related to the stage of selecting the problem's single solution.

For the first typical signal and the first method of the basic SLAE construction, the system (28) under $n1 = 3$, $n2 = 1$ is written as follows:

$$\sum_{i=1}^{6} A_{ki} X_i = B_k,\ k = 1, 2, \ldots, 6$$

$$A_{ki} = \begin{cases} \int_{t_k}^{t_{k+1}} \tau^{i-1} u(\tau) d\tau, & i \le n1 = 3 \\ -\dfrac{1}{i-n1}(t_{k+1}^{i-n1} - t_k^{i-n1}), & i > n1 \end{cases}$$

$$B_k = u(t_k) - u(t_{k+1}).$$

In this case, the following relationship between the source $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$, $\tilde{\beta}_1$ and intermediate $\gamma_j = X_{j+n1} = X_{3+j}$ unknown variables is observed:

$$\tilde{\alpha}_i = X_i,\ i = 1, 2, 3,\ \ X_4 = \gamma_1 = \tilde{\alpha}_1\tilde{\beta}_1,\quad X_5 = \gamma_2 = \tilde{\alpha}_2\tilde{\beta}_1,\quad X_6 = \gamma_3 = \tilde{\alpha}_3\tilde{\beta}_1.$$

Therefore, for determining $\tilde{\beta}_1$ variable, we have got three equations, of which the last two are unnecessary and can be discarded, so $\tilde{\beta}_1 = X_4/\tilde{\alpha}_1 = X_4/X_1$.

For comparison, Table 9 also provides outcomes of the sought-for $X(t)$, $a(t)$ process recovery, where $n1 = 4$, i.e., at approximating $\tilde{a}(t)$ estimate by a third-degree polynomial. These outcomes are contained in the fourth and fifth rows of Table 9, the distribution of partial intervals in which corresponds to the rule described here above.

As follows from the values of relative errors submitted in Table 9 as a percentage, the recovery of the first standard signal when excluding the influence of parametric effects is effective enough. The comparison of outcomes submitted in the second and third rows of table 9 with the appropriate outcomes presented in the fourth and fifth rows indicates that increasing the degree of the approximating polynomial for estimating $\tilde{a}(t)$ enables us, if necessary, to achieve a further reduction in errors in the measured signal recovery .

The last column of Table 9 provides the number of conditionality of the appropriate basic SLAE, which indicate the instability of the specified systems. Although the instability of SLAE's obtained by solving inverse problems of system dynamics represents a very widespread phenomenon, it requires special consideration each time, which will be examined in the next chapter.

In conclusion, let us consider the issue that is very important in the practice of measuring the fleeting process parameters - namely, the issue concerning the correction of the measurement system's dynamic properties. The essence of the systems' dynamic property correction, in particular of the measurement systems, is that the so-called corrective network is connected to this inertial system, which results in the situation when the combined system speed (this system + the corrective network) proves to be significantly higher compared to this system speed. Note, however, that correction of the system dynamic properties has not found any significant application in the theory and practice of dynamic measurements of non-electric quantities, since creation of corrective networks requires application of the information that is difficult of attainment in this field of measurement, namely, the information regarding the values of the parameters of the MS itself under actual measurement conditions. As far as non-stationary systems for measuring non-electric quantities are concerned, attempts to apply conventional methods of correction of the measurement system dynamic properties, seem to be ill-fated in this case.

In light of the above, let us focus again on the data presented in Table 9 for the first standard signal. Based on the data contained in the first row, it follows that if measured signal recovery $X(t)$ is limited only to implementation of the invariance algorithm on the first interval, error of $X(t)$ signal recovery is less than 0.06 %, and $T01$ duration of the recovery process is 0.1 sec.

At the same time, direct calculations demonstrate that the relative deviation of $u(t)$ readings of the measurement system from $X(t)$ measured signal reaches a value of ~ 0.06% at $t \approx 4$ sec without using the invariance algorithm. That is, in this particular measurement mode, application of the invariance algorithm outcomes, theoretically speaking leads to ~ a 40-fold increase in the speed of the process of producing the sought-for measurement information. It is clear that reducing $T01$ length of the first interval of the invariance algorithm implementation enables us to obtain a further increase in the speed of the process of obtaining measurement information. As far as the time spent on calculations during the invariance algorithm implementation is concerned, it is usually negligible.

All the above stated represents the property of the invariance principle, which is single-channel from a physical standpoint. This is essentially a correction of the dynamic properties of this inertial measurement system.

Recovery of the second standard signal

Outcomes of simulating the process of $X(t)$ measured signal recovery through the invariance algorithm for the second standard signal, i.e. the signal described in time by the linear law, are stated here below:

$$N2 = 2; \quad X(t) = \beta_1 + \beta_2 t.$$

Herewith, the law of variation in time of $a(t)$ parameter is also assumed to be linear one:

$$N1 = 2; \quad a(t) = \alpha_1 + \alpha_2 t.$$

The specified outcomes are presented in table 10 with the corresponding Appendix, in which the following values are taken as constant quantities α_1, α_2, β_1, β_2:

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \beta_1 = 2, \quad \beta_2 = 40.$$

The appendix to table 10, relating to the last (fifth) row of this table, submits absolute values of $u(t)$ readings of the measurement system, the restored $X(t)$ signal, and $\tilde{X}(t)$ estimates obtained as a result of the invariance algorithm implementation, for comparison.

Estimates $\tilde{a}(t)$, $\tilde{X}(t)$ are approximated by the same algebraic polynomials as in the first standard signal case.

Table 10. **Recovery of the second standard signal**

P	$t_1 \le t \le t_1 + T0P$	TP	$T0P$	Indirect criteria		Direct criteria		cond 1(A) / cond 2(A)
				$\rho I_{i\,\text{от}}$	$\rho I_{u\,\text{от}}$	$\rho a_{\text{от}}$	$\rho X_{\text{от}}$	
1	$0.04 \le t \le 0.14$	0.02	0.1	$4.277 \cdot 10^{-9}$	$4.148 \cdot 10^{-9}$	$4.209 \cdot 10^{-8}$	$4.614 \cdot 10^{-8}$	8.83 •106 / 5.1 •106
2	$0.04 \le t \le 0.54$	0.1	0.5	$1.409 \cdot 10^{-10}$	$1.183 \cdot 10^{-10}$	$2.089 \cdot 10^{-10}$	$2.952 \cdot 10^{-10}$	8.65 •104 / 7.1 •104
3	$0.04 \le t \le 2.54$	0.5	2.5	$1.62 \cdot 10^{-12}$	$4.423 \cdot 10^{-12}$	$2.5 \cdot 10^{-12}$	$4.9 \cdot 10^{-12}$	4.27 •104 / 2.8 •104
4	$0.04 \le t \le 12.54$	2.5	12.5	$4.135 \cdot 10^{-6}$	$7.215 \cdot 10^{-8}$	$3.11 \cdot 10^{-8}$	$7.156 \cdot 10^{-8}$	5.532 •106 / 2.8 •106
5	$0.04 \le t \le 62.54$	12.5	62.5	0.497	$3.752 \cdot 10^{-4}$	$1.9 \cdot 10^{-4}$	$3.75 \cdot 10^{-4}$	6.247 •109 / 3.27 •109

Appendix to table 10 ($p = 5$, $T05 = 62.5$)

t_i	$a(t)$	$\tilde{a}(t)$	$X(t)$	$\tilde{X}(t)$	$u(t)$
0.04	1.08	1.08	3.6	3.6	0.115
12.54	26.08	26.08	503.6	503.598	502.062
25.04	51.08	51.08	1003.6	1003.596	1002.816
37.54	76.08	76.08	1503.6	1503.594	1503.074
50.04	101.08	101.08	2003.6	2003.592	2003.204
62.54	126.08	126.08	2503.6	2503.591	2503.283

Selection of the only solution from among a variety of possible solutions is executed in the same way as before: the basic group of pairs (n^*1, n^*2) is selected from outcomes of the invariance algorithm implementation for different combinations of $n1$, $n2$ variables, which is followed by selection of a single pair from the basic group ($\bar{n}^*1$, $\bar{n}^*2$). Therefore, table 10 does not submit the process for the single solution selection. Distribution of partial intervals in the recovery of the second standard signal, as well as in the recovery of the first standard signal, corresponds to the rule described above.

Selection of a single solution demonstrates that $\bar{n}^*1 = N1 = 2$, $\bar{n}^*2 = N2 = 2$, so the basic SLAE at $n1 = 2$, $n2 = 2$, pursuant to (28), is written as:

$$\sum_{i=1}^{5} A_{ki} X_i = B_k, \quad k = 1, \ldots, 5$$

$$A_{ki} = \begin{cases} \int_{t_k}^{t_{k+1}} \tau^{i-1} u(\tau) d\tau, & i \le n_1 = 2 \\ -\frac{1}{i-n_1}(t_{k+1}^{i-n_1} - t_k^{i-n_1}), & i > n_1 \end{cases}$$

$$B_k = u(t_k) - u(t_{k+1}).$$

The relationship between the source $\tilde{\alpha}_1$, $\tilde{\alpha}_2$,$\cdot\tilde{\beta}_1$,$\cdot\tilde{\beta}_2$ and intermediate $\gamma_j = X_{n1+j} = X_{2+j}$ unknown variables in this case is written as:

$$\tilde{\alpha}_1 = X_1, \quad \tilde{\alpha}_2 = X_2, \quad \gamma_1 = \tilde{\alpha}_1\tilde{\beta}_1 \Rightarrow \tilde{\beta}_1 = X_3/X_1, \quad \gamma_2 = \tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1, \quad \gamma_3 = \tilde{\alpha}_2\tilde{\beta}_2.$$

As is obvious, we have two equations to determine the remaining unknown variable $\tilde{\beta}_2$. The second equation, as a superfluous one, can be discarded. So:

$$\tilde{\beta}_2 = \frac{X_4 - X_2 \frac{X_3}{X_1}}{X_1}.$$

Note, by the way, that the two given equations for determining $\cdot\tilde{\beta}_2$ variable in the entire range of $X(t)$ signal recovery, produce the outcomes that coincide with a very high degree of accuracy.

The structure of Table 10 and the meaning of the variables submitted in it coincide with those obtained for Table 9, relating to the first standard signal.

As follows from the data submitted in table 10, in the entire studied range $0.04 \le t \le 62.54$ of the second standard signal recovery, the invariance algorithm which excludes the impact of parametric effects, is characterized by a very high degree of accuracy.

Outcomes of determining the estimate of the second typical signal $\tilde{X}(t)$ at the first interval of the invariance algorithm implementation enables us, as in the case of the first typical signal, to increase significantly the speed of the process of obtaining the sought-for measurement information. Indeed, even on the first interval $[0.04 \le t \le 0.14]$, the invariance algorithm enables us to recover the sought-for parameters with an error of $\sim 5 \cdot 10^{-8}$ %, while $u(t)$ readings of the measurement system, which are usually the basis for estimating the measured signal variable as follows from the Appendix, still lag behind the appropriate value of the measured signal $X(t)$ by 0.012 % even at $t = 62.54$ sec. This indicates that application of the invariance algorithm leads to a ~1000-fold increase in the speed of the process of obtaining the sought-for measurement information.

In conclusion, let us once again consider the issue of observing monotonous change in the measured signal on the sections, the so-called “dynamically steady-state stage” of measurement. Clause 2.2, when stating the content of the invariance principle which is single-channel from physical standpoint, specified that application of this principle on the intervals entirely located in the zone of a dynamically steady state is illegal, since in the specified zone the equation describing the measurement system dynamic properties differs from the original “input–output” model of this MS, pursuant to which the invariance algorithm is built. The above-described rule for distribution of partial intervals was proposed for the purpose of this difficulty overcoming. Pursuant to this rule, for any $[t_1, t_1 + T0i]$ recovery interval of the measured signal, the first partial interval covers the measurement initial transition stage, which is the most informative in terms of the measurement system dynamic properties manifestation.

Recovery of the second standard signal through the invariance algorithm is a visual illustration of both the appearance of the problem in question and overcoming it. As follows from the data contained in the Appendix to table 10, at $t \ge 12.54$, the relative deviation between the MS readings and $X(t)$ measured signal remains virtually unchanged over time, and makes < 0.3 %, which means the MS “entry” into the “dynamically steady-state stage” of measurement. Note that for the outcomes of the invariance algorithm implementation related to the last row in Table 10, the duration of the segment [12.54; 62.54] of the steady-state measurement stage included in the general recovery interval [0.04; 62.54] is ~ 4 times longer compared to the transition process duration. Nevertheless, the accepted rule of partial interval distribution proceeds to ensure the invariance algorithm's effectiveness, and therefore the single-channel invariance principle validity.

Recovery of the third standard signal

The exceptional role of harmonic signals in the theory and practice of systems, and in particular measurement systems, is well known. In connection with this, as well as with regard to the fact that the invariance algorithm implementation for these signals' recovery has some distinctive peculiarities compared to the previous two types of signals, the outcomes related to the third standard signal recovery are considered more thoroughly.

We will suppose that the sought-for measured signal $X(t)$ and parameter $a(t)$ of the measurement system is described by the following expressions:

$$\tilde{X}(t) = \tilde{X}_0 + \tilde{A}_x \sin \omega t, \quad \tilde{a}(t) = \tilde{a}_0 + \tilde{A}_a \sin \omega t, \tag{29}$$

where X_0, A_x, a_0, A_a, w are unknown constant values.

In this case, $\tilde{X}(t)$, $\tilde{a}(t)$ estimates of the sought-for $X(t)$, $a(t)$ should also naturally be sought in the form of harmonic functions:

$$\tilde{X}(t) = \tilde{X}_0 + \tilde{A}_x \sin \omega t, \quad \tilde{a}(t) = \tilde{a}_0 + \tilde{A}_a \sin \omega t, \tag{30}$$

where $\tilde{X}_0$, $\tilde{A}_x$, $\tilde{a}_0$, $\tilde{A}_a$, ω are unknown estimates of the mean values, amplitudes and frequencies of $X(t)$ measured signal and $a(t)$ parameter, respectively.

In this case the residual in the integral form will be written as follows

$$I_{\text{int}}(t) = u(t) - u(t_1) + \int_{t_1}^{t} \left[\tilde{a}_0 + \tilde{A}_a \sin \omega\tau \right] u(\tau) d\tau -$$

$$\int_{t_1}^{t} \left[\tilde{a}_0 + \tilde{A}_a \sin \omega\tau \right] \cdot \left[\tilde{X}_0 + \tilde{A}_x \sin \omega\tau \right] d\tau.$$

After executing the possible integration, we get

$$I_{\text{int}}(t) = u(t) - u(t_1) + \left[\int_{t_1}^{t} u(\tau) d\tau \right] \tilde{a}_0 + \left[\int_{t_1}^{t} u(\tau) \sin \omega\tau \, d\tau \right] \tilde{A}_a -$$

$$[t - t_1] \, \tilde{a}_0 \tilde{X}_0 + \left[\frac{1}{\omega} (\cos \omega t - \cos \omega t_1) \right] (\tilde{a}_0 \tilde{A}_x + \tilde{A}_a \tilde{X}_0) -$$

$$\frac{1}{2} \left[(t - t_1) - \frac{1}{2\omega} (\sin 2\omega t - \sin 2\omega t_1) \right] \tilde{A}_a \tilde{A}_x. \tag{31}$$

This residual is nonlinear with respect to the unknown estimates $\tilde{a}_0$, $\tilde{A}_a$, $\tilde{X}_0$, $\tilde{A}_x$. For its linearization with respect to these unknown variables, it is necessary to introduce for consideration the generalized (intermediate) unknown variables. After the introduction of generalized unknown variables and a single notation for all unknown variables, we have the following:

$$X_1 = \tilde{a}_0, \quad X_2 = \tilde{A}_a, \quad X_3 = \tilde{a}_0 \tilde{X}_0, \quad X_4 = \tilde{a}_0 \tilde{A}_x + \tilde{A}_a \tilde{X}_0, \quad X_5 = \tilde{A}_a \tilde{A}_x.$$

As earlier, we denote the number of source unknown constant values included in the estimates of $\tilde{a}(t)$ parameter and $\tilde{X}(t)$ signal, respectively, by $n1$, $n2$. Then $n1 = 2$ (is $\tilde{a}_0$, $\tilde{A}_a$), $n2 = 2$ (is $\tilde{X}_0$, $\tilde{A}_x$). Suppose $n3$ is the number of generalized unknowns introduced with the aim of linearization of the residual, then $n3 = n1 + n2 - 1 = 3$. Total number of unknown variables is $n1 + n3 = 5$, i.e. is greater by $(n1-1) = 1$ compared to the source number of unknown variables $(n1 + n2) = 4$.

In the formulated statement of the problem related to the third standard signal recovery, $n1$, $n2$ variables are set at once. Due to the specified feature of the case under consideration, now the stage of selection of the only solution for the set problem from a variety of possible solutions will be as follows. First, the basic group (ω^*) of frequency values ω for oscillation of estimates $\tilde{a}(t)$, $\tilde{X}(t)$ are selected from outcomes of the invariance algorithm implementations based on the indirect criteria values corresponding to different values of frequency ω. Then, a single value $\bar{\omega}^*$ of frequency, which is characterized by the lowest values of indirect criteria, is selected from the basic group of frequencies (ω^*). The selected value $\bar{\omega}^*$ of frequency is adopted as the estimate of the sought-for w frequency, and $\tilde{a}_0$, $\tilde{A}_a$, $\tilde{X}_0$, $\tilde{A}_x$ variables corresponding to $\bar{\omega}^*$ frequency found in this process are taken as estimates of source unknown quantities $a0$, Aa, $X0$, Ax.

The following question arises: in what range of ω frequency variation is it necessary to search for $\bar{\omega}^*$ single frequency and therefore, the only solution to the set problem?

Usually ω frequency variation range, provided that it's quite narrow, is successfully selected, based on the fact that the estimated w frequency of $X(t)$ measured signal oscillation is conditioned, as a rule, by some known peculiarity of the basic process execution in the object under study. If the specified information regarding the object under study is not available, then one of the important conclusions of considering this non-stationary MS dynamics, formulated in clause 1.4, can be used. Namely, if $X(t)$ input signal and $a(t)$ parameter are changed pursuant to the harmonic law, the monochromatic oscillation of $X(t)$ signal at the MS input penetrates into the oscillation spectrum at the MS output, the basic frequency in which coincides with w frequency of the input signal, and the rest of the oscillation spectrum frequencies at the MS output are multiples of w frequency. Therefore, analysis of $u(t)$ output signal spectrum enables to determine a certain range of changes in ω value, in which a single value of $\bar{\omega}^*$ frequency should be searched and, therefore, the only solution to the set problem, based on the spectrum basic frequency variable value.

The basic SLAE of the invariance algorithm for the third typical signal, constructed by means of the first method taking into consideration the type of residual (31), will be written as follows

$$\left[\int_{t_k}^{t_{k+1}} u(\tau)d\tau\right] X_1 + \left[\int_{t_1}^{t_{k+1}} u(\tau)\sin\omega\tau\, d\tau\right] X_2 -$$

$$[t_{k+1} - t_k]\, X_3 + \left[\frac{1}{\omega}(\cos\omega t_{k+1} - \cos\omega t_k)\right] X_4 -$$

$$\frac{1}{2}\left[(t_{k+1} - t_k) - \frac{1}{2\omega}(\sin 2\omega t_{k+1} - \sin 2\omega t_k)\right] X_5 = u(t_k) - u(t_{k+1}), \tag{32}$$

$$k = 1, \ldots, 5,\ t_k = t_1 + T(k-1),\quad t_{k+1} = t_1 + Tk,\quad T = \frac{T0}{n1+n3} = \frac{T0}{5}.$$

where $T0$ is the length of the sought-for $a(t)$, $X(t)$ processes recovery.

We introduce the following notation for this system' components:

$$A_{k1} = \int_{t_k}^{t_{k+1}} u(\tau)d\tau;\quad A_{k2} = \int_{t_k}^{t_{k+1}} u(\tau)\sin\omega\tau d\tau;\quad A_{k3} = -(t_{k+1} - t_k) = -T$$

$$A_{k4} = \frac{1}{\omega}(\cos\omega t_{k+1} - \cos\omega t_k);$$

$$A_{k5} = -\frac{1}{2}\left[(t_{k+1} - t_k) - \frac{1}{2\omega}(\sin 2\omega t_{k+1} - \sin 2\omega t_k)\right]$$

$$B_k = u(t_k) - u(t_{k+1}),\quad k = 1, 2, \ldots, (n1 + n3) = 5.$$

Then the basic SLAE will be written in standard form:

$$\sum_{i=1}^{n_1+n_3} A_{ki} X_i = B_k,\ k = 1, \ldots, 5.$$

Having solved this system and found X_i, $i = 1, \ldots, 5$, then we determine the sought-for estimates $\tilde{a}_i(t)$, $\tilde{X}(t)$, taking into account that:

$$\tilde{a}_0 = X_1,\quad \tilde{A}_a = X_2,\quad \tilde{X}_0 = X_3/X_1,\quad \tilde{A}_x = (X_4 - \tilde{A}_a\tilde{X}_0)/\tilde{a}_0.$$

Herewith, the value of the fifth found unknown variable X_5 remains unused. It is clear that if X_5variable is used to determine $\tilde{A}_x$, then the same outcome will be obtained as in the previous case (the difference is at the level of calculation error).

Table 11 submits outcomes of simulation of the recovery process for the third standard signal through the invariance algorithm for the recovery interval [0.04; 0.14]. Herewith, the following are taken as the values of the source constants included in $a(t)$, $X(t)$ expression:

$$a_0 = 10,\quad A_a = 8,\quad X_0 = 100,\quad A_x = 40;$$

the cyclic frequency $w = 2\pi F = 6.283185$, which corresponds to the frequency $F = 1$ Hz.

Appendix to table 11 submits outcomes of the invariance algorithm implementation at recovering the sought-for $a(t)$, $X(t)$ for one specific frequency value ω, which enables to compare the absolute values of the sought-for and obtained estimates for them.

Table 11. **Recovery of the third standard signal (t_1 = 0,04; $T0$ = 0.1)**

ω	Indirect criteria		Direct criteria	
	$\rho I_{i\ \text{от}}$	$\rho I_{u\ \text{от}}$	$\rho a_{\text{от}}$	$\rho X_{\text{от}}$
6.27637	0.213	0.151	0.474	0.168
6.282	$3.715 \cdot 10^{-2}$	$2.63 \cdot 10^{-2}$	$8.219 \cdot 10^{-2}$	$2.933 \cdot 10^{-2}$
6.283	$5.812 \cdot 10^{-3}$	$4.114 \cdot 10^{-3}$	$1.284 \cdot 10^{-2}$	$4.59 \cdot 10^{-3}$
6.283065	$3.773 \cdot 10^{-3}$	$2.671 \cdot 10^{-3}$	$8.334 \cdot 10^{-3}$	$2.98 \cdot 10^{-3}$
6.283125	$1.892 \cdot 10^{-3}$	$1.339 \cdot 10^{-3}$	$4.179 \cdot 10^{-3}$	$1.494 \cdot 10^{-3}$
6.283185	$1.114 \cdot 10^{-10}$	$7.885 \cdot 10^{-11}$	$4.072 \cdot 10^{-10}$	$9.91 \cdot 10^{-11}$
6.283245	$1.872 \cdot 10^{-3}$	$1.326 \cdot 10^{-3}$	$4.136 \cdot 10^{-3}$	$1.479 \cdot 10^{-3}$
6.28365	$1.458 \cdot 10^{-2}$	$1.032 \cdot 10^{-2}$	$3.219 \cdot 10^{-2}$	$1.152 \cdot 10^{-2}$
6.28437	$3.719 \cdot 10^{-2}$	$2.633 \cdot 10^{-2}$	$8.204 \cdot 10^{-2}$	$2.939 \cdot 10^{-2}$
6.290	0.2146	0.1522	0.470	0.170

Appendix to table 11

$\omega = 6{,}29$; $\tilde{a}_0 = 9{,}930473$; $\tilde{A}_a = 7{,}997586$; $\tilde{X}_0 = 100{,}701184$; $\tilde{A}_x = 38{,}484971$

t	$a(t)$	$\tilde{a}(t)$	$X(t)$	$\tilde{X}(t)$	$u(t)$
0.04	11.99	11.922	109.948	110.282	37.559
0.06	12.945	12.878	114.725	114.883	54.097
0.08	13.854	13.787	119.27	119.26	68.924
0.1	14.702	14.636	123.511	123.343	81.999
0.12	15.476	15.41	127.382	127.069	93.36
0.14	16.164	16.098	130.821	130.378	103.096

As follows from the data submitted in table 11, the indirect criteria variables are so sensitive to frequency variations that both the selection of the basic group (ω^*) of frequencies and the selection of a single frequency ($\bar{\omega}^*$) from this group prove to be very simple. Indeed, since the indirect criterion value ($\rho I_{i\ \text{от}}$ or $\rho I_{u\ \text{от}}$) for a certain frequency proves to be at least seven orders of magnitude less than the values of this criterion for all other frequencies, then, obviously, the frequency which corresponds to this minimum value of the indirect criterion should be taken as the only sought-for frequency $\bar{\omega}^*$. Based on the data presented in the table, it follows that $\bar{\omega}^* = 6.283185$ is the specified frequency, which almost coincides with the source constant value w, and the corresponding errors in the $a(t)$, $X(t)$ processes recovery are negligible.

Appendix to table 11 specifically submits the data related to the frequency $\omega = 6.29$, which is one of the boundaries of ω frequency variation interval, i.e. the data which represents one of the worst of the invariance algorithm implementations. As can be seen, even for this case, $\tilde{X}(t)$ estimate of the measured signal obtained through the invariance algorithm is quite satisfactorily consistent with the measured signal $X(t)$ itself, while $u(t)$ readings of the measurement system are very far from the true nature of the measured signal variation.

Thus, the above stated enables to conclude that recovery of the third standard signal through the invariance algorithm, excluding the impact of parametric effects, is executed with a very high accuracy degree.

When the input signal $X(t)$ changes periodically, $u(t)$ readings of the non-stationary measurement system are more complex, however regular as well. Therefore, no stage of monotonous approximation of MS readings to the measured signal is stipulated for the third standard signal, and the measurement system itself is always in a state of full and active manifestation of its dynamic properties. In other words, unlike the first and second standard signals, the problem related to the MS “entry” into such “dynamically steady-state stage” of measurements, in which the measurement system dynamic properties are not manifested in full extent, becomes irrelevant for the third standard signal, since at this measurement stage the MS dynamic properties are described by more simple equations compared to the source “input–output” model of the measurement system.

The above stated enables to assume that application of the previously formulated rule for the partial interval distribution is possible for the third standard signal, but it's not obligatory.

As for the third standard signal, $[t_1, t_1 + T0]$ interval of the sought-for $a(t)$, $X(t)$ processes recovery may include the initial stage, i.e. transient, measurement stage (table 11), but it may not include, either. Moreover, $[t_1, t_1 + T0]$ recovery interval can be located, generally speaking, on any portion of time axis, which should not significantly affect the quality of the sought-for processes recovery. The stated facts significantly simplify the process of the third standard signal recovery through the invariance algorithm.

Table 12 submits the outcomes of simulating the recovery process for the third standard signal through the invariance algorithm, in which the recovery intervals [t_1, t_1 + $T0P$] do not contain the measurement's initial stage, and are located on different portions of the time axis.

Table 12. **Recovery of the third standard signal at different locations of the recovery interval**

P	$t_1 \leq t \leq t_1 + T0P$	TP	$T0P$	Indirect criteria		Direct criteria		cond 1(A) cond 2(A)
				$\rho I_{i\ от}$	$\rho I_{u\ от}$	$\rho a_{от}$	$\rho X_{от}$	
1	$3 \leq t \leq 3.5$	0.1	0.5	$1.093 \cdot 10^{-4}$	$2.534 \cdot 10^{-5}$	$1.575 \cdot 10^{-5}$	$2.651 \cdot 10^{-5}$	1.4 •104 7.7 •103
2	$10 \leq t \leq 10.5$	0.1	0.5	$5.729 \cdot 10^{-5}$	$5.875 \cdot 10^{-6}$	$6.889 \cdot 10^{-6}$	$6.474 \cdot 10^{-6}$	1.4 •104 7.7 •103
3	$15 \leq t \leq 15.5$	0.1	0.5	$1.901 \cdot 10^{-3}$	$4.96 \cdot 10^{-4}$	$6.327 \cdot 10^{-4}$	$5.157 \cdot 10^{-4}$	1.4 •104 7.7 •103
4	$24 \leq t \leq 24.5$	0.1	0.5	$1.482 \cdot 10^{-3}$	$3.128 \cdot 10^{-4}$	$6.904 \cdot 10^{-5}$	$3.117 \cdot 10^{-4}$	1.4 •104 7.7 •103
5	$60 \leq t \leq 60.5$	0.1	0.5	$5.956 \cdot 10^{-4}$	$2.258 \cdot 10^{-5}$	$5.033 \cdot 10^{-6}$	$2.249 \cdot 10^{-5}$	1.4 •104 7.7 •103

The stage of selection of the only solution for the set problem from a set of possible solutions, i.e. the stage of determining the single frequency, is not submitted in table 12, since it is already reflected in table 11; the data provided in table 12 are related to $\bar{\omega}^* = 6.283185$ frequency.

As follows from the above stated simulation outcomes, the recovery of the sought-for $a(t)$, $X(t)$ at each of [t_1, t_1 + + $T0P$] intervals of the study range [3; 60.5] is executed with a high degree of accuracy, which confirms the above-mentioned peculiarities of the third standard signal recovery. As for the differences observed, pursuant to table 12, in the error values for the sought-for process recovery during transition from one interval to another, these differences are not of any regular character, so they can be ignored because of their small size.

Note that in case of a certain ratio between the oscillation period of the sought-for variables, and TP duration of the partial interval, the basic SLAE matrix for the third standard signal may be singular. It is clear that these cases should be excluded from consideration as unacceptable ones.

Let us complete our analysis of the simulation process related to the third standard signal by the above stated information. In the more general case of periodic oscillations of the measured signal $X(t)$ and $a(t)$ parameter, the invariance algorithm can be constructed on the basis of an approximation of the corresponding estimates through trigonometric polynomials.

Outcomes of the model implementation of the invariance principle which is a single-channel from the physical standpoint submitted in this chapter, enable us to make the following conclusion:

The invariance principle which is single-channel from a physical standpoint, meaningfully representing the formulation and solution of the extended dynamic measurement objective, is a fairly effective means of excluding the impact of parametric effects on the dynamic measurement accuracy under conditions of deterministic variations in time of the measured signal and the measurement system parameters.

This principle content, along with the solution to the basic problem – the exclusion of the impact of parametric effects, also simultaneously enables us, directly in the measurement process, to solve the problem related to the parametric identification of linear non-stationary measurement system.

In addition, the implementation of the single-channel invariance principle is accompanied by a correction of the measurement system's dynamic properties, which leads to an increase in the speed of the process of obtaining measurement information by several orders of magnitude.

Some generalizations of the outcomes obtained so far in the areas of extending the class of both measurement systems and measured signals are submitted in the following chapters.

CHAPTER 4

SPECIAL ISSUES RELATED TO THE APPLICATION OF THE PRINCIPLE OF INVARIANCE

4.1. ON THE STABILITY OF INVARIANCE ALGORITHMS

As we would expect in solving the inverse measurement problem, the basic SLAEs for invariance algorithms, regardless of the methods used for their compilation, are significantly unstable, and hence, the algorithm for recovering the desired signal $X(t)$ and the parameter $a(t)$ is also unstable.

Attempts to increase the stability of the invariance algorithm by direct use of general regularization methods - method of regularization by A. N. Tikhonov [13], including using extended matrices [7] and regularization methods based on singular decomposition [14], does not always lead to an acceptable result: sometimes the required increase in stability is achieved only with an unacceptable increase in the recovery errors of the signal $X(t)$ and the parameter $a(t)$.

In this connection, along with the use of these common methods of regularization, researchers often turn to some private methods of increasing stability in solving inverse problems. These private techniques are associated, as a rule, with the specific content of the problem to be solved or with the features of the model of the object under study.

To achieve a significant increase in the stability of an invariance algorithm with the preservation of an acceptable number of errors of signal recovery $X(t)$ and parameter $a(t)$ is possible reception, the essence of which is the use of derivatives of some order of this residual, along with the residual referred to in this paragraph $R(t)$. As a result, it is possible to move from the original basic SLAE AX = B of the invariance algorithm to the aggregate of some SLAE of a lower order AOX = B0, and several independent equations.

Let us describe the specific content of the given reception in relation to the three model signals considered earlier, and to the second method of compiling the main SLAE, although neither the type of signal, nor the method of obtaining the main SLAE, generally speaking, have any fundamental importance in the application of this technique.

First typical signal

For the first typical signal and its evaluation we have:

$$a(t) = a_к - (a_к - a_н)e^{\mu t}, \quad \mu = \text{const}, \quad X(t) = \beta_1 = \text{const},$$

$$\tilde{a}(t) = \sum_{j=1}^{n1} \tilde{\alpha}_j t^{j-1}, \quad n1 = 3, \quad \tilde{X}(t) = \tilde{\beta}_1, \quad n2 = 1.$$

Consider the case when the structure $\tilde{a}(t)$ and the fact that the unknown $X(t)$ - a constant value, have already been set according to the previously described scheme.

Overall residual form $R(t)$:

$$R(t) = u(t)\,\tilde{a}(t) - \tilde{X}(t)\cdot\tilde{a}(t) + v(t), \quad v(t) = \frac{du(t)}{dt} \tag{1}$$

takes the form of

$$R(t) = u(t)\,(\tilde{\alpha}_1 + \tilde{\alpha}_2 t + \tilde{\alpha}_3 t^2) - \tilde{\beta}_1\tilde{\alpha}_1 - \tilde{\beta}_1\tilde{\alpha}_2 t - \tilde{\beta}_1\tilde{\alpha}_3 t^2 + v(t) \tag{2}$$

or

$$R(t) = \sum_{i=1}^{3}[u(t)t^{i-1}]X_i - \sum_{i=4}^{6}[t^{i-4}]X_i + v(t). \tag{3}$$

Applying the second method of compiling the basic SLAE to this residual expression results in the next basic SLAE of the sixth order

$$\sum_{i=1}^{6} A_{ki} X_i = B_k,\ k = 1, \dots, 6, \quad T = T0/5, \quad t_k = t_1 + T(k-1),$$

$$B_k = -v(t_k), \quad i = 1, \dots, 6$$

$$A_{ki} = \begin{cases} u(t_k) t_k^{i-1}, & i \le 3 \\ -t_k^{i-4}, & i > 3 \end{cases} \tag{4}$$

where $X_1 = \tilde{\alpha}_1$; $X_2 = \tilde{\alpha}_2$; $X_3 = \tilde{\alpha}_3$; $X_4 = \tilde{\alpha}_1\tilde{\beta}_1 \Rightarrow \tilde{\beta}_1 = X_4/X_1$; $X_5 = \tilde{\alpha}_2\tilde{\beta}_1$; $X_6 = \tilde{\alpha}_3\tilde{\beta}_1$.

Let us turn to the design of a system that has a smaller order than the basic SLAE.

For the first three derivatives of residuals we have:

$$\frac{dR(t)}{dt} = v(t)\ \tilde{\alpha}_1 + [v(t) \cdot t + u(t)]\ \tilde{\alpha}_2 + [v(t) \cdot t^2 + $$
$$+ 2u(t) \cdot t] \tilde{\alpha}_3 - \tilde{\beta}_1 \tilde{\alpha}_2 - 2\tilde{\beta}_1 \tilde{\alpha}_3 t + w_0(t) \tag{5}$$

$$\frac{d^2R(t)}{dt^2} = w_0(t)\ \tilde{\alpha}_1 + [w_0(t) \cdot t + 2v(t)]\ \tilde{\alpha}_2 + [w_0(t) \cdot t^2 + $$
$$+ 4v(t) \cdot t + 2u(t)]\ \tilde{\alpha}_3 - 2\tilde{\beta}_1 \tilde{\alpha}_3 + h(t) \tag{6}$$

$$\frac{d^3R(t)}{dt^3} = h(t)\ \tilde{\alpha}_1 + [h(t) \cdot t + 3w_0(t)]\ \tilde{\alpha}_2 + [h(t) \cdot t^2 + $$
$$+ 6w_0(t) \cdot t + 6v(t)]\ \tilde{\alpha}_3 + p(t), \tag{7}$$

where $w_0(t) = \dfrac{d^2u(t)}{dt^2}$, $h(t) = \dfrac{d^3u(t)}{dt^3}$, $p(t) = \dfrac{d^4u(t)}{dt^4}$.

Let us pay attention to the structure of each of these expressions.

If we enter a new unknown $\tilde{\beta}_1\tilde{\alpha}_2 = \gamma_1$, $\tilde{\beta}_1\tilde{\alpha}_3 = \gamma_2$ in the expression (5) and require zero of this expression in the points $k = 1, \dots, M$, $M \ge 5$, $t_k = t_1 + T(k-1)$, $T = T0/(M-1)$, we get a new SLAE of the fifth order, from which we can find all the required values $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$, $\tilde{\beta}_1$. At the same time, one of the two expressions $\gamma_1 = \tilde{\beta}_1\tilde{\alpha}_2$, $\gamma_2 = \tilde{\beta}_1\tilde{\alpha}_3$ to determine the parameter $\tilde{\beta}_1$ is superfluous, and we do not use the basic condition for the residuals, namely $R(t_k) = 0$.

If we enter a new unknown $\tilde{\beta}_1\tilde{\alpha}_3 = \gamma_2$ in the expression (6) and demand zero of this expression in the points $k = 1, \dots, M$, $M \ge 4$, $T = T0/(M-1)$, $t_k = t_1 + T(k-1)$, we get a new SLAE of the fourth order, from which we can find all the required values $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$, $\tilde{\beta}_1$.

If we demand zero of the expression (7) in points $k = 1, \dots, M$, $M \ge 3$, $T = T0/(M-1)$, $t_k = t_1 + T(k-1)$, we get a new SLAE of the third order, from which we can find only the sought $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$ and the sought $\tilde{\beta}_1$ which can be found from one of the expressions (2), (5), (6), equating this expression to zero at any point t_k of the corresponding recovery interval.

Naturally, we refer to the third of these possibilities, namely, to use the third order system to determine $\tilde{\alpha}_i(t)$, $i = 1, 2, 3$:

$$\sum_{i=1}^{3} A0_{ki} \tilde{\alpha}_i = B0_k,\ k = 1, \dots, M,\ M \ge 3,\ T = T0/(M-1),\ t_k = t_1 + T(k-1),$$

$$A0_{k1} = h(t_k), \quad A0_{k2} = t_k \cdot h(t_k) + 3w_0(t_k),$$

$$A0_{k3} = t_k^2 \cdot h(t_k) + 6w_0(t_k) \cdot t_k + 6v(t_k), \quad B0_k = -p(t_k), \tag{8}$$

and then the original, primary, residuals expression (2) to determine the value of $\tilde{\beta}_1$. In this case, for unknown $\tilde{\beta}_1$ we will have:

$$\tilde{\beta}_1 = u(t) + \frac{v(t)}{\tilde{a}(t)}, \tag{9}$$

where t takes any value from $t_k = t_1 + T(k-1)$, $k = 1, \dots, 6$, $T = T0/5$.

So, from the point of view of solving the problem of restoring the desired signal $X(t)$ and parameter $a(t)$, the aggregate of the third order system (8) and the equation $R(t_k) = 0$, that is, the expression (9), is equivalent to the initial basic system (4) of the sixth order.

Therefore, now the stability of the invariance algorithm is determined by the stability of the third order system (8), and the error of determining я $\tilde{\beta}_1$ by the formula (9) is almost at the level of errors in the definition of parameter $\tilde{a}(t)$.

Table 1 presents the results of the invariance algorithm implementation for the first typical signal and the second method of obtaining the main SLAE in three variants:

Table 1. **Increasing the stability of the recovery algorithm of the first typical signal**

P	$t_1 \le t \le t_1 + T0P$	$T0P$	Indirect Criteria $\rho I_{u\ \text{от}}$	Direct Criteria		cond 1 (A) cond$_s$ (A) cond 1 ($A0$)
				$\rho a_{\text{от}}$	$\rho X_{\text{от}}$	
1	$0.04 \le t \le 0.34$	0.3	$2.903 \cdot 10^{-3}$ 0.248 $1.435 \cdot 10^{-3}$	$7.1 \cdot 10^{-2}$ 4.341 0.054	$6.4 \cdot 10^{-2}$ 4.177 0.048	$8.3 \cdot 10^{9}$ 366 $\sim 10^{4}$
2	$0.04 \le t \le 1.54$	1.5	$9.015 \cdot 10^{-3}$ 0.986 $7{,}75 \cdot 10^{-3}$	$6.5 \cdot 10^{-2}$ 5.882 0.062	$4.1 \cdot 10^{-2}$ 4.169 0.038	$1.3 \cdot 10^{7}$ 378 352
3	$0.04 \le t \le 7.54$	7.5	$1.8 \cdot 10^{-3}$ 2.177 0.011	$8.2 \cdot 10^{-2}$ 76.195 0.032	$2.9 \cdot 10^{-3}$ 5.907 0.012	$7.3 \cdot 10^{5}$ $7{,}6 \cdot 10^{3}$ $1.5 \cdot 10^{3}$
4	$0.04 \le t \le 15.04$	15	$3.522 \cdot 10^{-4}$ $4.934 \cdot 10^{4}$ $9.228 \cdot 10^{-3}$	0.235 154.81 0.86	$4.458 \cdot 10^{-5}$ 1.94 0.011	$2.8 \cdot 10^{7}$ $3 \cdot 10^{4}$ 330

– use for recovery of signal $X(t)$ and parameter $a(t)$ of the main SLAE - system of the sixth order (4);

– use for restoring the signal $X(t)$ and parameter $a(t)$ of a singular decomposition, which is applied directly to a system (4). Of the six singular numbers s_i of matrix A, only the first four are left, and the fifth and sixth singular numbers are equal to zero. The conditionality number of matrix A is calculated by the formula cond$_s$ (A) = s_1/s_4, where s_1, s_4 – the largest and smallest singular numbers (left);

– use to restore the desired $X(t)$, $a(t)$ system of the third order (8) and equation (9) at $t = t_1$.

With any $p = 1, ..., 4$, the sequence number of the row of numbers in Table 1 corresponds to the sequence number of the options described above. Thus, in any p, the first line refers to the results of the recovery of the search obtained by the first of the above options, and the third line refers to the results obtained by the third of the described options.

Referring directly to the results presented in Table 1 (relative values are given in percentages), and based on generally accepted assumptions about the sustainability of systems, we conclude that the main SLAE (4) (first lines at $p = 1, ..., 4$) is obviously unstable, as the number of conditionality of this system at the recovery interval of the sought value from $[t_1, t_1 + 0.3]$ to $[t_1, t_1 + 15]$ varies from cond1 (A) = $7.3 \cdot 10^5$ to cond1 (A) = $8.3 \cdot 10^9$.

Further, the use of the required singular decomposition of matrix A gives results (second lines at $p = 1, ..., 4$), which are satisfactory in terms of stability in the entire range recovery, but not satisfactory in accuracy at intervals $0.04 \le t \le 7.54$ ($p = 3$), $0.04 \le t \le 15.04$ ($p = 4$); the increase in the number of equations in SLAE (4) (number of points M) does not improve the results recovery of the sought values.

Finally, the use of the retrieval system (8) and the expression (9) (third lines at $p = 1, ..., 4$) gives results that are in the stability zone and acceptable in accuracy over the entire range recovery sought (the number of conditionality was reduced to $\sim 10^6$ times with $p = 1$).

Note that instead of a recovery algorithm based on (8) - (9), we can build a recovery algorithm based on second-order SLAE to define $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and two expressions to define $\tilde{\alpha}_3$, $\tilde{\beta}_1$. This allows us to reduce the number of conditionalities by at least an order of magnitude, maintaining an acceptable level of errors in recovery of the sought value, which requires the use of the derivative of the following order.

Thus, the transition from the main SLAE of the sixth order (4) to the SLAE of the third order (8) and the ratio (9) allows us to solve the problem of making the algorithm invariance of a steady character.

Second typical signal

For the second typical signal and its evaluation we have:

$$a(t) = \sum_{j=1}^{N1} \alpha_j t^{j-1}, \ N1 = 2, \quad X(t) = \sum_{j=1}^{N2} \beta_j t^{j-1}, \ N2 = 2,$$

$$\tilde{a}(t) = \sum_{j=1}^{n1} \tilde{\alpha}_j t^{j-1}, \ n1 = 2, \quad \tilde{X}(t) = \sum_{j=1}^{n2} \tilde{\beta}_j t^{j-1}, \ n2 = 2.$$

As follows from the above expressions, we believe that the stage of searching for values $n1$, $n2$ (the number of unknown constant values in expressions for $\tilde{\alpha}(t)$ and $\tilde{X}(t)$) has already passed, and now it is necessary to refer to the stable invariance algorithm.

Instead of the previous residuals (2) and derivatives of first-third orders of residuals (5) - (7) in this case we have:

$$R(t) = u(t)\cdot\tilde{\alpha}_1 + t\cdot u(t)\cdot\tilde{\alpha}_2 - \tilde{\alpha}_1\tilde{\beta}_1 - t(\tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1) - t^2(\tilde{\alpha}_2\tilde{\beta}_2) + v(t) \tag{2a}$$

$$\frac{dR(t)}{dt} = v(t)\,\tilde{\alpha}_1 + [t\cdot v(t) + u(t)]\,\tilde{\alpha}_2 - (\tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1) - 2t(\tilde{\alpha}_2\tilde{\beta}_2) + w_0(t) \tag{5a}$$

$$\frac{d^2R(t)}{dt^2} = w_0(t)\,\tilde{\alpha}_1 + [t\cdot w_0(t) + 2v(t)]\,\tilde{\alpha}_2 - 2(\tilde{\alpha}_2\tilde{\beta}_2) + h(t) \tag{6a}$$

$$\frac{d^3R(t)}{dt^3} = h(t)\,\tilde{\alpha}_1 + [t\cdot h(t) + 3w_0(t)]\,\tilde{\alpha}_2 + p(t) \tag{7a}$$

Here, the notation has the same meaning as for the first typical signal. The application of the second method of compiling the main SLAE to the residuals expression (2a) gives the main SLAE in the form of:

$$\sum_{i=1}^{5} A_{ki} X_i = B_k,\ k = 1, \ldots, 5,\quad T = T0/4,\quad t_k = t_1 + T(k-1),$$
$$B_k = -v(t_k),$$
$$A_{ki} = \begin{cases} t_k^{i-1}\cdot u(t_k), & i \le 2 \\ (-1)t_k^{i-3}, & i > 2 \end{cases} \tag{10}$$

The relationship between the original and the new unknown is as follows:

$X_1 = \tilde{\alpha}_1$; $X_2 = \tilde{\alpha}_2$; $X_3 = \tilde{\alpha}_1\tilde{\beta}_1 \Rightarrow \tilde{\beta}_1 = X_3/X_1$; $X_4 = (\tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1)$; $X_5 = \tilde{\alpha}_2\tilde{\beta}_2$.

Analyzing the expression structures of residual derivatives (5a) - (7a), we notice that using the expression of the third derivative residue (7a), we can find unknown $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, for which it is sufficient to require that this expression be equal to zero at points $k = 1, \ldots, M \ge 2$, $t_k = t_1 + T(k-1)$, $T = T0/(M-1)$. This results in a second order system:

$$\sum_{i=1}^{2} A0_{ki}\tilde{\alpha}_i = B0_k,\quad k = 1, \ldots,\quad M \ge 2,\quad B0_k = -p(t_k), \tag{11}$$

$$i = 1, 2,\quad A0_{k1} = h(t_k),\quad A0_{k2} = h(t_k)\cdot t_k + 3w_0(t_k).$$

It remains to find unknowns $\tilde{\beta}_1$ and $\tilde{\beta}_2$.

The value of $\tilde{\beta}_2$ is found using the expression for the derivative of the second order of the residuals, from the equation $\frac{d^2R(t)}{dt^2} = 0,\ t = t_k$. It gives:

$$\tilde{\beta}_2 = \frac{1}{2\tilde{\alpha}_2}\cdot\{w_0(t)\,\tilde{\alpha}_1 + [t\cdot w_0(t) + 2v(t)]\,\tilde{\alpha}_2 + h(t)\},\ \ t = t_k. \tag{12}$$

Finally, the value $\tilde{\beta}_1$ can be found using either the expression of the first derivative residual (5a) or the expression of the residual itself (2a). Using the second option, we have

$$\tilde{\beta}_1 = u(t) - t\,\tilde{\beta}_2 + \frac{v(t)}{\tilde{\alpha}_1 + \tilde{\alpha}_2\cdot t},\quad t = t_k. \tag{12a}$$

Thus, the implementation of the invariance algorithm based on the basic SLAE of the fifth order (10) was reduced to the implementation of this algorithm based on the system of the second order (11) and two equations:

$$\left.\frac{d^2R(t)}{dt^2}\right|_{t=t_k} = 0,\quad R(t_k) = 0. \tag{13}$$

Therefore, the stability of the invariance algorithm is now determined by the stability of the second order system (11).

Table 2 presents the results of the invariance algorithm implementation for the second typical signal and the second method of compiling the basis of SLAE at $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 2$, $\beta_2 = 40$. The upper figures refer to the case of the use of the SLAE framework of the fifth order (10), and the lower figures refer to the use of the second order system (11) and equations (13). Since the recovery range of the sought is quite wide ($0.24 \le T0 \le 61.44$), when using the second-order system (11), 16 points were taken (M = 16), and the solution to this system was found using a singular expansion.

Table 2. **Increasing the stability of the recovery algorithm of the second typical signal**

P	$t_1 \le t \le t_1 + T0P$	$T0P$	Indirect Criteria	Direct Criteria		cond 1 (A) cond$_s$ ($A0$)
			$\rho I_{u\,\text{от}}$	$\rho a_{\text{от}}$	$\rho X_{\text{от}}$	
1	$0.04 \le t \le 0.28$	0.24	$6.1\cdot10^{-10}$ $2.6\cdot10^{-6}$	$2.3\cdot10^{-9}$ $9.9\cdot10^{-6}$	$2.3\cdot10^{-9}$ $3.1\cdot10^{-6}$	$1.6\cdot10^{5}$ 5.8
2	$0.04 \le t \le 1$	0.96	$2.2\cdot10^{-10}$ $1.1\cdot10^{-6}$	$2.2\cdot10^{-10}$ $1.8\cdot10^{-7}$	$3.6\cdot10^{-10}$ $1.1\cdot10^{-6}$	$1.4\cdot10^{4}$ 1.9
3	$0.04 \le t \le 3.88$	3.84	$2.2\cdot10^{-10}$ $2.3\cdot10^{-7}$	$2.1\cdot10^{-10}$ $6.4\cdot10^{-7}$	$2.3\cdot10^{-10}$ $3.2\cdot10^{-7}$	$6.9\cdot10^{4}$ 1.7
4	$0.04 \le t \le 15.4$	15.36	10^{-7} $6.2\cdot10^{-8}$	$3.9\cdot10^{-8}$ $1.2\cdot10^{-6}$	10^{-7} $7.6\cdot10^{-8}$	$2.5\cdot10^{6}$ 6
5	$0.04 \le t \le 61.48$	61.44	$3.8\cdot10^{-6}$ $4.9\cdot10^{-7}$	$1.3\cdot10^{-6}$ $8.8\cdot10^{-4}$	$3.8\cdot10^{-6}$ $2.5\cdot10^{-7}$	$1.3\cdot10^{8}$ 602

As can be seen from the results presented in Table 2, the invariance algorithm based on the use of the basic SLAE (10) is unstable, while the invariance algorithm based on the use of the second order system (11) and two equations (13), is quite stable. Compared with the case of the use of the basic SLAE, the number of conditionalities decreased to ~ $2\cdot10^5$ times.

By the way, if for the most severe conditions of recovery of signal $X(t)$ and parameter $a(t)$, namely, for p = 5 (the widest recovery interval) take M = 32, then the number of conditionalities for SLAE of the second order will decrease ~ 20 times and become only 30 (instead of 602).

The errors in recovering the signal $X(t)$ and the parameter $a(t)$ in the second order system (11) are negligible, as in the case of SLAE.

Third typical signal

For the third typical signal and its evaluation we have:

$$a(t) = a_0 + A_a \sin\, wt, \quad X(t) = X_0 + A_x \sin\, wt,$$

$$\tilde{a}(t) = \tilde{a}_0 + \tilde{A}_a \sin\,\omega t, \quad \tilde{X}(t) = \tilde{X}_0 + \tilde{A}_x \sin\,\omega t,$$

Consider the case when the cyclic frequencyω is already defined and $\omega = w$.

The expression for the residuals in this case takes the form:

$$R(t) = u(t)\cdot\tilde{a}_0 + [u(t)\sin\,\omega t]\cdot\tilde{A}_a - \tilde{X}_0\tilde{a}_0 - [\tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a]\cdot\sin\,\omega t - \left[\frac{1}{2}(1-\cos 2\omega t)\right]\cdot\tilde{A}_x\tilde{A}_a + v(t). \tag{2b}$$

Derivatives of first-third orders of residual are:

$$\frac{dR(t)}{dt} = v(t)\cdot\tilde{a}_0 + [v(t)\sin\,\omega t + u(t)\omega\cos\omega t]\cdot\tilde{A}_a - [\omega\sin\,2\omega t]\tilde{A}_x\tilde{A}_a - [\omega\cos\omega t](\tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a) + w_0(t) \tag{5b}$$

$$\frac{d^2R(t)}{dt^2} = w_0(t)\cdot\tilde{a}_0 + [w_0(t)\sin\,\omega t + 2v(t)\omega\cos\omega t - u(t)\omega^2\sin\,\omega t]\cdot\tilde{A}_a - [2\omega^2\cos 2\omega t]\tilde{A}_x\tilde{A}_a + [\omega^2\sin\,\omega t](\tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a) + h(t) \tag{6b}$$

$$\frac{d^3R(t)}{dt^3} = h(t)\cdot\tilde{a}_0 + [h(t)\sin\,\omega t + 3w_0(t)\omega\cos\omega t - 3v(t)\omega^2\sin\,\omega t - u(t)\omega^3\cos\omega t]\tilde{A}_a + [4\omega^3\sin\,2\omega t]\tilde{A}_x\tilde{A}_a + [\omega^3\cos\omega t](\tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a) + p(t) \tag{7b}$$

In the expressions (5b) - (7b), the function $w_0(t)$, as before, denotes the derivative function $v(t)$.

The application of the second method of compiling the basic SLAE to the residual expression (2b) gives the basic SLAE of the fifth order in the form of:

$$\sum_{i=1}^{5} A_{ki}X_i = B_k,\ k = 1, \ldots, 5, \quad T = T0/4, \quad t_k = t_1 + T(k-1),$$

$$B_k = -v(t_k),\quad i = 1, \ldots, 5,\quad A_{k1} = u(t_k),\ A_{k2} = u(t_k)\sin\omega t_k,$$

$$A_{k3} = -1,\ A_{k4} = -\sin\omega t_k,\quad A_{k5} = \frac{1}{2}(\cos 2\omega t_k - 1). \tag{14}$$

$$X_1 = \tilde{a}_0,\ X_2 = \tilde{A}_a,\ X_3 = \tilde{X}_0\tilde{a}_0 \Rightarrow \tilde{X}_0 = \frac{X_3}{X_1},\ X_4 = \tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a \Rightarrow$$

$$\tilde{A}_x = \frac{X_4 - \tilde{X}_0\tilde{A}_a}{X_1},\ X_5 = \tilde{A}_a\tilde{A}_x.$$

Analyzing the structures of expressions (5b) – (7b), we notice that unknown $\tilde{a}_0, \tilde{A}_a, \tilde{X}_0, \tilde{A}_x$ can be found using any of these expressions, requiring zero of the selected expression at points t_k, $k = 1, \ldots, M \geq 4$.
At the same time, for the purpose of linearization it would be necessary to introduce a new unknown $\gamma_1 = \tilde{A}_x\tilde{A}_a$, $\gamma_2 = \tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a$, that would lead to the system of the fourth order. However, it is advisable to carry out some transformations, bearing in mind the reduction of the order of SLAE, which will be the basis of the invariance algorithm.

We will require that each of the expressions (5b) – (7b) be equal to zero at the points $t = t_k$. Consider the obtained expressions for $\left.\frac{dR(t)}{dt}\right|_{t=t_k}$, $\left.\frac{d^2R(t)}{dt^2}\right|_{t=t_k}$ and exclude the unknown $(\tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a)$; then consider, for example, the obtained expressions for $\left.\frac{d^2R(t)}{dt^2}\right|_{t=t_k}$, $\left.\frac{d^3R(t)}{dt^3}\right|_{t=t_k}$ and exclude the same unknown $(\tilde{a}_0\tilde{A}_x + \tilde{X}_0\tilde{A}_a)$. As a result, omitting the indication of t_k, points, we get two new expressions:

$$\varphi_1(t)\tilde{a}_0 + \varphi_2(t)\tilde{A}_a + \varphi_3(t)\tilde{A}_a\tilde{A}_x + f(t) = 0$$
$$\varphi_1(t)\tilde{a}_0 + \varphi_2(t)\tilde{A}_a + \varphi_3(t)\tilde{A}_a\tilde{A}_x + f(t) = 0 \tag{15}$$

where:

$$\varphi_1(t) = v(t)\omega^2\sin\omega t + \omega\cos\omega t\cdot w_0(t)$$

$$\varphi_2(t) = \omega^2\sin^2\omega t\cdot v(t) + \frac{1}{2}\omega\sin 2\omega t\cdot w_0(t) + 2\omega^2\cos^2\omega t\cdot v(t)$$

$$\varphi_3(t) = \omega^3\sin\omega t\cdot\sin 2\omega t + 2\omega^3\cos\omega t\cdot\cos 2\omega t$$

$$f(t) = \omega^2\sin\omega t\cdot w_0(t) + \omega\cos\omega t\cdot h(t)$$

$$\psi_1(t) = \omega^3\cos\omega t\cdot w_0(t) - \omega^2\sin\omega t\cdot h(t)$$

$$\psi_2(t) = 2\omega^4\cos^2\omega t\cdot v(t) - \omega^2\sin^2\omega t\cdot h(t) - \omega^3\sin 2\omega t\cdot w_0(t) +$$
$$+ 3\omega^4\sin^2\omega t\cdot v(t)$$

$$\psi_3(t) = 2\omega^5\cos\omega t\cdot\cos 2\omega t + 4\omega^5\sin\omega t\cdot\sin 2\omega t$$

$$F(t) = \omega^3\cos\omega t\cdot h(t) - \omega^2\sin\omega t\cdot p(t)$$

Now we notice that each of the expressions (15) can be used as a basis for constructing SLAE of the third order to determine the search $\tilde{a}_0, \tilde{A}_a, \tilde{A}_x$, if we put $\tilde{A}_a\cdot\tilde{A}_x = \gamma$.

Stopping, for example, on the first of the expressions (15), to determine the search, we have the following SLAE of the third order:

$$\sum_{i=1}^{3} A0_{ki}X_i = B0_k,\quad k = 1, \ldots,\quad M \geq 3,\ T = T0/(M-1),$$
$$t_k = t_1 + T(k-1),\quad i = 1, \ldots, 3,$$
$$A0_{k1} = \varphi_1(t_k),\quad A0_{k2} = \varphi_2(t_k),\quad A0_{k3} = \varphi_3(t_k),\ B0_k = -f(t_k), \tag{16}$$
$$X_1 = \tilde{a}_0,\ X_2 = \tilde{A}_a,\ X_3 = \tilde{A}_a\tilde{A}_x \Rightarrow \tilde{A}_x = \frac{X_3}{X_2}.$$

The remaining unknown value of $\tilde{X}_0$ can be found from any of the expressions (2b), (5b) – (7b), requiring zero of the selected expression at any point $t_k \in [t_1, t_1 + T0]$. When using the residual expression (2b) for unknown $\tilde{X}_0$, we get

$$\widetilde{X}_0 = \frac{u(t)\widetilde{a}_0 + u(t)\sin\omega t \cdot \widetilde{A}_a - \widetilde{a}_0\widetilde{A}_x \sin\omega t - \frac{1}{2}(1-\cos 2\omega t)\widetilde{A}_x\widetilde{A}_a + v(t)}{\widetilde{a}_0 + \widetilde{A}_a \sin\omega t}. \quad (17)$$

So instead of the main system of the fifth order (14), now we have a system of the third order (16) to define $\widetilde{a}_0, \widetilde{A}_a, \widetilde{A}_x$ and one equation $R(t) = 0$, $t = t_k$ to determine $\widetilde{X}_0$ and the stability of the algorithm invariance is determined by the stability of the third order system (16).

Finally, if necessary, it is possible to come to a second-order system. Indeed, excluding the unknown expression (15) $\widetilde{A}_a \cdot \widetilde{A}_x$, we get the expression:

$$\varphi_1(t)\psi_3(t) - \varphi_3(t)\psi_1(t) \cdot \widetilde{a}_0 + [\varphi_2(t)\psi_3(t) - \varphi_3(t)\psi_2(t)\widetilde{A}_a =$$
$$= \varphi_3(t)F(t) - \psi_3(t)f(t)$$

Writing this ratio for the points t_k, $k = 1, \ldots, M \geq 2$, we get the SLAE of the second order. Having determined the unknown $\widetilde{a}_0, \widetilde{A}_a$, from this system $\widetilde{A}_x$ we find from any of the expressions (15) at an arbitrary point $t_k \in [t_1, t_1 + T0]$. The remaining unknown value $\widetilde{X}_0$ is found, as in the previous case, by the formula (17).

However, as will be seen from the following, there is no need to switch to a second order system for the third typical signal. Therefore, let us consider the implementation of the invariance algorithm for the third typical signal based on the use of the third order system (16) and one equation $R(t) = 0$.

Table 3 presents the results of the invariance algorithm implementation for the third typical signal and the second method of compiling the main SLAE at the following values of the parameters of the desired $X(t)$ and $a(t)$:

$$a_0 = 10, \quad A_a = 8, \quad X_0 = 100, \quad A_x = 40.$$

The results given in this table are obtained by the number of equations in the system of the third order (16) equal to 12 ($M = 12$), and the solution of this system was obtained by using singular decomposition. The upper figures in the table refer to the case when the invariance algorithm is implemented on the basis of the main system of the fifth order (14), the lower ones are based on the use of the third order system (16) and expression (17) to define $\widetilde{X}_0$.

Table 3. **Increase of stability of the algorithm of recovery of the third typical signal**

P	$t_1 \leq t \leq t_1 + T0P$	$T0P$	Indirect Criteria $\rho I_{u\ \text{от}}$	Direct Criteria $\rho a_{\text{от}}$	Direct Criteria $\rho X_{\text{от}}$	cond 1 (A) cond$_s$ ($A0$)
1	$0.04 \leq t \leq 0.16$	0.12	$1.,3\cdot10^{-10}$ $1.8\cdot10^{-10}$	$5.9\cdot10^{-11}$ $5.49\cdot10^{-10}$	$2\cdot10^{-10}$ $4.3\cdot10^{-10}$	$8\cdot10^{5}$ 116
2	$0.04 \leq t \leq 0.52$	0.48	10^{-13} $4\cdot10^{-5}$	$5\cdot10^{-13}$ $3.4\cdot10^{-5}$	10^{-13} $4.6\cdot10^{-5}$	$6.5\cdot10^{3}$ 112
3	$0.04 \leq t \leq 1.96$	1.92	$1.7\cdot10^{-9}$ $2\cdot10^{-2}$	$2.8\cdot10^{-9}$ $2\cdot10^{-2}$	$2.2\cdot10^{-9}$ $2\cdot10^{-2}$	$1.3\cdot10^{4}$ 232
4	$0.04 \leq t \leq 7.72$	7.68	$4.9\cdot10^{-7}$ $5.7\cdot10^{-4}$	$7.6\cdot10^{-7}$ $4.9\cdot10^{-4}$	$6.6\cdot10^{-7}$ $5\cdot10^{-4}$	$1.1\cdot10^{4}$ 117

As can be seen from the results presented in Table 3, the invariance algorithm implemented using the third order (16) system and one equation to determine the fourth unknown, is quite stable, and the errors in the recovery of the signal $X(t)$ and the parameter $a(t)$, as in the case of the use of the unstable basic SLAE of the fifth order, are negligible.

As follows from the above, the built stable algorithms of recovery of the second and third typical signals are based on the joint use of the proposed private method of increase of stability and the method of solving the resulting new SLAE using singular decomposition.

Above, in order to reduce the presentation, in the construction of stable algorithms it was assumed that the stage of selecting the structure of estimates $\widetilde{X}(t)$, $\widetilde{a}(t)$ was already completed, and only the task of increasing the stability of the algorithms was set. It is clear, however, that the creation of stable invariance algorithms, generally speaking, should precede the search for the structure of estimates $\widetilde{X}(t)$, $\widetilde{a}(t)$.

This is the content of one of the possible ways to increase the stability of invariance algorithms when restoring the measured signal $X(t)$.

4.2. INVARIANCE ALGORITHM FOR LINEAR MEASURING SYSTEMS WITH LUMPED UNSTRUCTURED PARAMETERS

The description of the dynamic property of measurement systems of non-electric quantities is usually based on the use of certain physical laws, which allows us to form a mathematical model of behavior of these systems, for example, in the form of specific ordinary differential equations, and these equations are often linear. The construction of invariance algorithms for these measuring systems of known structure, is devoted to all the previous presentation.

It should be noted, however, that there are systems that cannot be described by differential equations. Therefore, for the sake of generality, it is interesting to consider a situation where only the fact of its linearity is known about the measuring system, and there is no information about its structure.

From the general theory of systems, it is known that for any continuous linear systems with lumped parameters of an arbitrary structure with zero initial conditions, the integral relation holds:

$$u(t) = \int_0^t \tilde{w}(t,\tau) X(\tau) d\tau, \tag{18}$$

where: $u(t)$, $X(t)$ - output and input signals of the system, respectively; $\tilde{w}(t,\tau)$ - some generalized characteristic of the dynamic properties of the system, called impulse transient function of the system.

In different areas of dynamics of systems the content problem associated with the use of mathematical model is different (18).

In particular, if the input $X(t)$ and output $u(t)$ values of the dynamic system are known and it is necessary to find the characteristic $\tilde{w}(t,\tau)$ of the system, then this is the domain of the theory of identification of systems. If the output signal $u(t)$ of the system and the characteristic $\tilde{w}(t,\tau)$ are known, and it is necessary to find the input signal $X(t)$, then this is an area of the theory of dynamic measurements, bearing in mind the traditional formulation of problems of dynamic measurements.

Although a huge number of works [3, 5, 8, 15] are devoted to the analysis of model (18) in the two areas noted, intensive research on model (18) in these areas continues.

In accordance with the idea and principle of constructing an invariance algorithm, restoration of the measured signal $X(t)$ from the output signal $u(t)$ should be carried out according to an algorithm that is independent of the parameters of the measuring system. So far, in the study of measuring systems of a known structure, this goal has been achieved by the inclusion of time-variable parameters in the number of unknown parameters in the model of the measuring system.

Therefore now, by analogy, in the study of measuring systems of an arbitrary (unknown) structure, it is advisable to include among the unknowns in the model (18) the generalized characteristic of the system $\tilde{w}(t,\tau)$.

In other words, in this more general task of recovering the measured signal, as well as in considering linear MS's of a known structure, it is necessary to move into an extended measurement task.

So, for linear measuring systems (with lumped parameters) of an arbitrary structure as a mathematical model characterizing its dynamic properties, we now have a relationship (18), in which the output signal $u(t)$ is known, and the measured signal $X(t)$ and the characteristic are not known systems $\tilde{w}(t,\tau)$.

It is clear that in this formulation of the measurement problem, the ratio (18) cannot be attributed to any type of integral equations in the generally accepted sense of this concept. Therefore, the ratio (18) is further considered simply as a functional relationship, establishing the relationship between the input and output signals of the measuring system.

Not having in mind a detailed consideration here of the indicated general statement of the measurement problem with the aim of constructing an invariance algorithm, we show however, that such a statement of the problem makes sense, and under certain conditions, leads to the goal. Thus, we draw attention to some new questions that arise in this case.

As in the case of measurement systems of a known structure, the estimate of $\tilde{X}(t)$ of the measured signal $X(t)$ can be presented as an algebraic polynomial:

$$\tilde{X}(t) = \sum_{j=1}^{n2} \tilde{\beta}_j t^{j-1}. \tag{19}$$

The generalized characteristic $\tilde{w}(t,\tau)$ of the measuring system is a function of two variables t, τ and the simplest representation for this model can be the expression:

$$g(t,\tau) = g_1 + g_2 t + g_3 \tau, \tag{20}$$

where $g(t,\tau)$ – a model for the true but unknown impulse transient function $\tilde{w}(t,\tau)$ of the measuring system; g_1, g_2, g_3 – constant but unknown values.

The following simulation results take the form (20) of the function $g(t,\tau)$, i.e. the value $n1$ - the number of unknown constant values in the model $g(t,\tau)$ is fixed by the value $n1 = 3$.

The residual value, which characterizes the process of recovery of the measured signal $X(t)$, takes the form:

$$I_u(t) = u(t) - \int_0^t g(t,\tau) \cdot \widetilde{X}(\tau) d\tau. \tag{21}$$

It is easy to notice that after substituting expressions (19), (20) in (21), we can enter the intermediate unknown X_i, $i = 1, 2, \ldots$ and, using any of the methods described earlier, get a system of linear algebraic equations relative to the intermediate unknown X_i, $i = 1, 2, \ldots$ However, the transition from intermediate unknown to initial unknown g_1, g_2, g_3, $\tilde{\beta}_1$, …, $\tilde{\beta}_n$ can generally be performed only by solving a nonlinear system of algebraic equations. Generally speaking, the described procedure can be considered as one of the possible directions of constructing an invariance algorithm for the measuring systems of arbitrary structure.

Since in this work we limit ourselves to the use of only linear systems of algebraic equations when constructing the invariance algorithm, we will consider a different way of constructing algorithms' invariance for the measuring systems of an arbitrary structure.

This objective can be achieved, for example, either by selecting a special type of model $g(t,\tau)$, or by modifying the original generic model (18). On the second of these directions we will stop.

Along with the general initial model (18), we will consider the functional ratio:

$$u(t) + \int_0^t \widetilde{w}(t,\tau)\psi(\tau)d\tau = \int_0^t \widetilde{w}(t,\tau)X(\tau)d\tau, \tag{22}$$

where $\psi(\tau)$ – some known function.

Assume the recovery of the measured signal be carried out at some time interval $[t_1, t_1 + T0]$, on which the condition is fulfilled:

$$|\Psi(t)| << |X(t)|. \tag{23}$$

Next, suppose that the problems described by the relationship (18) and (22) have solutions that allow us to find the original unknown values. Then, given the condition (23), it can be expected that the two objectives will be close to each other. At the same time, the stronger the inequality is performed (23), the closer the solution to the problem (22) is to the solution of the basic initial problem (18). Thus, under these assumptions, it is possible to solve the auxiliary problem (22) and take this decision as an approximate solution to the main problem (18).

Obviously, there are different kinds of functions $\Psi(t)$, satisfying the condition (23). One of these dependencies can be the following:

$$\Psi(t) = u(t). \tag{24}$$

The logical basis for this choice is that for stable linear measuring systems with zero initial conditions, it is possible to select such a narrow recovery interval of the measured signal, that on this interval the condition will be met:

$$|u(t)| << |X(t)|,$$

and hence the condition (23).

It is assumed that the beginning of the recovery process $t = t_1$ of the signal is close enough to the beginning of the measurement process t = 0.

Thus, along with the general model (18) of the linear measuring system, an auxiliary model is considered, described by the functional ratio of:

$$u(t) = \int_0^t \widetilde{w}(t,\tau)X(\tau)d\tau - \int_0^t \widetilde{w}(t,\tau)u(\tau)d\tau. \tag{25}$$

Note that the described approach to obtaining the solution to the initial problem by transition with a particular purpose to some auxiliary problem, is quite often used in the theory of integral equations, for example, when regularizing the Volterra integral equation of the first kind [8]. The essence of this regularization is as follows.

Let the integral Volterra equation of the first kind be given:

$$\int_0^x K(x,t)q(t)dt = f(x),$$

where $K(x,t)$ – the known core of the integral equation; $f(x)$ the known function; $q(t)$ – the unknown function.

The equation has an unstable solution. Along with this equation the auxiliary integral equation with a positive parameter is considered α:

$$\alpha q(x) + \int_0^x K(x,t)q(t)dt = f(x).$$

The latter equation is the Volterra equation of the second kind, and its solution is stable.

It is proved that at the small parameter α and fulfillment of certain conditions imposed on the function $K(x,t)$, the solution of the auxiliary equation is close to the solution of this initial equation. Therefore, as an approximate solution to this integral Volterra equation of the first kind, a stable solution to the auxiliary integral equation of the second kind of Volterra is taken.

Thus, the addition to this equation of the small term $\alpha q(x)$ has a very specific purpose - to use instead of the unstable solution to this equation close to it, the steady solution to the auxiliary equation. This approach to this initial task is often quite effective.

In our case, the introduction of an additional term pursues a completely different purpose, namely, the construction of an invariance algorithm, the implementation of which uses only linear systems of algebraic equations. Therefore, it is natural that the content of the additional term in the statement of the problem of constructing the invariance algorithm is completely different than in the Volterra integral equation discussed above.

Let us return to the construction of the invariance algorithm, turning to the ratio (25).

As an illustration of the applicability of the described approach, consider the basic fragments of constructing the invariance algorithm, and the process of recovery of the measured signal for the three previously considered typical signals. In order to reduce the presentation, let us assume that in all three cases the variables of the measuring system change uniformly over time and therefore the generalized characteristic $\tilde{w}(t,\tau)$ of the measuring system is the same for the three recuperations of the measured signal $X(t)$.

Therefore, the process of constructing an invariance algorithm and recovering the measured $X(t)$ signal for three specific cases follows:

for the first typical signal

$$X(t) = \beta_1 = \text{const};$$

for the second typical signal

$$X(t) = \beta_1 + \beta_2 t, \quad \beta_1 = \text{const}, \quad \beta_2 = \text{const};$$

for the third typical signal

$$X(t) = X_0 + A_x \sin wt, \; X_0 = \text{const}, \quad A_x = \text{const}, \quad w = \text{const}.$$

In the simulation, a first-order linear system was used as a measuring system described by the equation

$$\frac{du(t)}{dt} + a(t) \cdot u(t) = a(t) \cdot X(t),$$

and for parameter $a(t)$ in all three cases a linear dependency was assumed:

$$a(t) = \alpha_1 + \alpha_2 t, \quad \alpha_1 = \text{const}, \quad \alpha_2 = \text{const}.$$

Of course, the specified input data about the input signals $X(t)$, the parameter $a(t)$, and the structure of the true measuring system, as it should be, are not used in the process of constructing the algorithm invariance, and recovery of the measured signal $X(t)$. Further, in the modeling process we do not look for the best approximation for the impulse transient function $\tilde{w}(t,\tau)$, but as its model we take the simplest of possible representations, namely, representation $g(t,\tau)$ as (20).

It is clear that with this simple representation $g(t,\tau)$ it is not expected that, along with the high-precision recovery of the measured signal $X(t)$, it will be possible to recover simultaneously with high accuracy and impulse transient function $\tilde{w}(t,\tau)$ of the measuring system. The desire to improve the accuracy of recovery and the impulse transient function $\tilde{w}(t,\tau)$ would require the complication of this model $g(t,\tau)$, for example by adding (20) members with higher degrees to the submission variables t, τ. At the same time, as elsewhere, from the results of invariance algorithm implementation relating to different values $n1$, $n2$ – the number of unknown constant values in estimates $g(t, \tau)$, $\tilde{X}(t)$ respectively, it would be necessary to define the main group of pairs (n^*1, n^*2) from which then we would select the only acceptable pair ($\bar{n}^*1$, $\bar{n}^*2$), which we will not do here, limiting only to case $n1 = 3$.

Finally, we note that in presenting the simulation results below, as we have done on several occasions before, we omit the statement of the preliminary phase of recovery of the measured signal, i.e. the stage of setting the input signal type. Thus, the following recovery results of the measured signal $X(t)$ are already at the stage of determining the dependence on the duration of $T0$ of the recovery interval $[t_1, t_1 + T0]$ numeric characteristics of the evaluation parameters $\tilde{X}(t)$ of the measured signal $X(t)$, for which the functional law of change over time has already been established.

First typical signal

For this signal as an estimate of $\tilde{X}(t)$ of the measured signal $X(t)$, we have $\tilde{X} = \tilde{\beta}_1$, and as a model $g(t,\tau)$ of the impulse transient function $\tilde{w}(t,\tau)$ we take representation (20). Substituting these values in the ratio (25) instead $X(t)$ and $\tilde{w}(t,\tau)$, we get the expression for the residuals:

$$I(t) = u(t) + \int_0^t g(t,\tau)u(\tau)\,d\tau - \int_0^t g(t,\tau)\tilde{X}(\tau)d\tau,$$

or after transformations:

$$I(t) = u(t) + \left[\int_0^t u(\tau)\,d\tau\right]g_1 + \left[t\int_0^t u(\tau)d\tau\right]g_2 + \left[\int_0^t \tau\, u(\tau)\,d\tau\right]g_3 -$$

$$tg_1\tilde{\beta}_1 - t^2(g_2\tilde{\beta}_1 + \frac{1}{2}g_3\tilde{\beta}_1).$$

By introducing the consideration of the intermediate sought value:

$$tg_1\tilde{\beta}_1 - t^2(g_2\tilde{\beta}_1 + \frac{1}{2}g_3\tilde{\beta}_1).$$

we have:

$$I(t) = u(t) + \left[\int_0^t u(\tau)\,d\tau\right]X_1 + \left[t\int_0^t u(\tau)d\tau\right]X_2 + \left[\int_0^t \tau\, u(\tau)\,d\tau\right]X_3 - tX_4 - t^2X_5. \quad (26)$$

Finally, using the second method of compiling the main system, we get the necessary SLAE:

(27)

where $B_k = u(t_k)$, $A_{k1} = -\int_0^{t_k} u(\tau)\,d\tau$, $A_{k2} = -t_k\int_0^{t_k} u(\tau)\,d\tau$, $A_{k3} = -\int_0^{t_k} \tau\cdot u(\tau)\,d\tau$, $A_{k4} = t_k$, $A_{k5} = t_k^2$.

Here t_1 и $T0$ – the beginning and length of the recovery interval of the $X(t)$ signal, respectively.

After the solution of the system (27), the initial required values are determined from the ratios:

$$g_1 = X_1,\ g_2 = X_2,\ g_3 = X_3,\ \tilde{\beta}_1 = \frac{X_4}{g_1}.$$

An expression $X_5 = g_2\tilde{\beta}_1 + \frac{1}{2}g_3\tilde{\beta}_1$, from which we can also find

value $\tilde{\beta}_1$, is superfluous, it can be discarded; of course, it gives the same numeric value of value $\tilde{\beta}_1$, as the first expression for $\tilde{\beta}_1$.

Below is Table 4, of the recovery of the first typical signal using the SLAE-based invariance algorithm (27). This table refers to the case when the numerical values of the parameters in the input data in the simulation are taken:

$$\beta_1 = 100;\quad \alpha_1 = 0.1;\quad \alpha_2 = 0.2.$$

The criteria $\rho X_{от}$ and $\rho I_{u\ от}$ in this table have meaning:

$$\rho X_{от} = \frac{\int_{t_1}^{t_1+T0} \left|X(t) - \tilde{X}(t)\right|dt}{\int_{t_1}^{t_1+T0} \left|X(t)\right|dt} \quad -$$

direct criterion for estimating the error of recovery of signal $X(t)$;

$$\rho Iu_{от} = \frac{\int_{t_1}^{t_1+T0} \left|u(t) - U(t)\right|dt}{\int_{t_1}^{t_1+T0} \left|u(t)\right|dt} \quad -$$

indirect criterion in which the signal appears as an estimate of $U(t)$ of the output signal $u(t)$:

$$U(t) = \int_0^t g(t,\tau)\,\tilde{X}(\tau)d\tau.$$

Note: function $I(t)$ was used in the construction of the system (27), which is a residual for the auxiliary problem (25). The function $I_u(t)$ is a non-viscous initial problem (18) and importantly, when using the criterion $\rho I_{u\ \text{от}}$, unlike previously considered systems of a given structure, now when considering systems of an arbitrary structure, there is no need to know a particular type of function $u(t)$ as a solution to the corresponding differential equation with variable coefficients.

As it follows from the data given in Table 4, for the first typical signal the problem of recovery of the measured signal invariant to parametric phenomena is solved quite effectively, and for the case of a measuring system of arbitrary (unknown) structure.

Table 4. **Implementation of the invariance algorithm for the first typical signal in the unknown structure of the measuring system**

P	$t_1 \le t \le t_1 + T0$	$T0P$	Indirect Criteria $\rho I_{u\ \text{от}}$	Direct Criterion $\rho X_{\text{от}}$	Restored Signal $\tilde{X}(t) = \tilde{\beta}_1$	cond 1 (A)
1	[0.04; 0.09]	0.05	0.367	$3.6 \cdot 10^{-5}$	100.000036	$4.5 \cdot 10^{9}$
2	[0.04; 0.14]	0.1	0.554	$2.4 \cdot 10^{-6}$	100.000002	$2.7 \cdot 10^{8}$
3	[0.04; 0.24]	0.2	0.985	$1.4 \cdot 10^{-7}$	99.999999	$1.6 \cdot 10^{7}$
4	[0.04; 0.34]	0.3	1.48	$4.4 \cdot 10^{-8}$	99.999999	$4.6 \cdot 10^{6}$
5	[0.04; 0.44]	0.4	2.036	$3.4 \cdot 10^{-9}$	99.999999	$1.9 \cdot 10^{6}$
6	[0.04; 0.54]	0.5	2.654	$4.1 \cdot 10^{-10}$	100.000000	$9.7 \cdot 10^{5}$
7	[0.04; 0.64]	0.6	3.332	$5.8 \cdot 10^{-10}$	99.999999	$6.3 \cdot 10^{5}$
8	[0.04; 0.74]	0.7	4.073	$1.004 \cdot 10^{-10}$	100.000000	$4.3 \cdot 10^{5}$
9	[0.04; 0.84]	0.8	4.875	$4.86 \cdot 10^{-11}$	100.000000	$3.1 \cdot 10^{5}$
10	[0.04; 0.94]	0.9	5.741	$1.45 \cdot 10^{-10}$	100.000000	$2.3 \cdot 10^{5}$
11	[0.04; 1.04]	1.0	6.67	$4.1 \cdot 10^{-10}$	100.000000	$1.76 \cdot 10^{5}$

Second typical signal

Substituting in (25) instead of $\tilde{w}(t,\tau)$ and $X(t)$ the expression (20) for the model of the impulse transient function $g(t,\tau)$ and the expression for the model of the measured signal

$$\tilde{X}(t) = \tilde{\beta}_1 + \tilde{\beta}_2 t,$$

we get the residuals expression for the second typical signal in the form:

$$I(t) = u(t) + \left[\int_0^t u(\tau)\,d\tau\right] g_1 + \left[t\int_0^t u(\tau)d\tau\right] g_2 + \left[\int_0^t \tau\, u(\tau)\,d\tau\right] g_3 -$$
$$t\,(g_1\tilde{\beta}_1) - t^2(g_2\tilde{\beta}_1 + \frac{1}{2}g_3\tilde{\beta}_1 + \frac{1}{2}g_1\tilde{\beta}_2) - t^3(\frac{1}{2}g_2\tilde{\beta}_2 + \frac{1}{3}g_3\tilde{\beta}_2).$$

Introduces the intermediate search for:

$$X_1 = g_1,\ X_2 = g_2,\ X_3 = g_3,\ X_4 = g_1\tilde{\beta}_1,\ X_5 = g_2\tilde{\beta}_1 + \frac{1}{2}g_3\tilde{\beta}_1 + \frac{1}{2}g_1\tilde{\beta}_2,$$
$$X_6 = \frac{1}{2}g_2\tilde{\beta}_2 + \frac{1}{3}g_3\tilde{\beta}_2.$$

Then the residual will take the form:

$$I(t) = u(t) + \left[\int_0^t u(\tau)\,d\tau\right] X_1 + \left[t\int_0^t u(\tau)d\tau\right] X_2 + \left[\int_0^t \tau\, u(\tau)\,d\tau\right] X_3 -$$
$$tX_4 - t^2X_5 - t^3X_6. \qquad (28)$$

Applying the second method of compiling the basic SLAE, we get:

$$\sum_{i=1}^{6} A_{ki}X_i = B_k, \quad k = 1, \ldots, 6,\ T = T0/5,\ t_k = t_1 + T(k-1), \qquad (29)$$

where $B_k = u(t_k)$, $A_{k1} = -\int_0^{t_k} u(\tau)\, d\tau$, $A_{k2} = -t_k \int_0^{t_k} u(\tau)\, d\tau$, $A_{k3} = -\int_0^{t_k} \tau \cdot u(\tau)\, d\tau$, $A_{k4} = t_k$, $A_{k5} = t_k^2$, $A_{k6} = t_k^3$.

After the solution of the system (29) the initial unknown values are determined from the ratios:

$$g_1 = X_1,\; g_2 = X_2,\; g_3 = X_3,\; \tilde{\beta}_1 = \frac{X_4}{g_1},\; \tilde{\beta}_2 = \frac{2X_5 - 2X_2\tilde{\beta}_1 - X_3\tilde{\beta}_1}{X_1}.$$

The expression $X_6 = \frac{1}{2} g_2 \tilde{\beta}_2 + \frac{1}{3} g_3 \tilde{\beta}_2$, from which the value of $\tilde{\beta}_2$ can also be determined is superfluous; after the decision of the system (29), the expression may be left unconsidered.

Table 5 shows the results of recovery of the second typical signal using a SLAE-based (29) invariance algorithm. In this case, the numerical values of the parameters in the input data during the simulation are taken:

$$\beta_1 = 2;\quad \beta_2 = 40;\quad \alpha_1 = 0.1;\quad \alpha_2 = 0.2.$$

The results shown in Table 5 show that for measuring systems of unknown structure, the recovery of the second typical signal with the help of the invariance algorithm is equally effective, as in the case of restoration of the first typical signal.

Table 5. **Implementation of the invariance algorithm for the second typical signal in the unknown structure of the measuring system**

P	$t_1 \le t \le t_1 + T0$	$T0P$	Indirect criteria $\rho I_{u\,от}$	Direct Criterion $\rho X_{от}$	Restored Signal		cond 1 (A)
					$\tilde{\beta}_1$	$\tilde{\beta}_2$	
1	[0.04; 0.09]	0.05	0.322	$2.4 \cdot 10^{-4}$	1.999972	39.999829	$4 \cdot 10^{12}$
2	[0.04; 0.14]	0.1	0.479	$2.18 \cdot 10^{-5}$	1.999999	39.999995	$1.2 \cdot 10^{11}$
3	[0.04; 0.24]	0.2	0.835	$2.07 \cdot 10^{-7}$	1.999999	39.999999	$3.6 \cdot 10^{9}$
4	[0.04; 0.34]	0.3	1.242	$1.63 \cdot 10^{-8}$	1.999999	39.999999	$4.4 \cdot 10^{8}$
5	[0.04; 0.44]	0.4	1.701	$1.33 \cdot 10^{-8}$	2.000000	40.000000	$9.7 \cdot 10^{7}$
6	[0.04; 0.54]	0.5	2.213	$5.1 \cdot 10^{-9}$	2.000000	40.000000	$2.955 \cdot 10^{7}$
7	[0.04; 0.64]	0.6	2.78	$3.3 \cdot 10^{-9}$	2.000000	39.999999	$1.1 \cdot 10^{7}$
8	[0.04; 0.74]	0.7	3.396	$1.06 \cdot 10^{-9}$	2.000000	40.000000	$4.8 \cdot 10^{6}$
9	[0.04; 0.84]	0.8	4.069	$1.43 \cdot 10^{-10}$	2.000000	40.000000	$2.3 \cdot 10^{6}$
10	[0.04; 0.94]	0.9	4.797	$1.13 \cdot 10^{-10}$	2.000000	40.000000	$1.2 \cdot 10^{6}$
11	[0.04; 1.04]	1.0	5.581	$2.59 \cdot 10^{-10}$	2.000000	40.000000	$7.45 \cdot 10^{6}$

Note that similar results are obtained when modeling the process of recovery of the measured signal using the invariance algorithm, if this signal is described by the algebraic polynomial higher degree.

Third typical signal

Substituting in (25) instead of $\tilde{w}(t,\tau)$ and $X(t)$ the expression (20) for the model of the impulse transient function $g(t,\tau)$ and the expression for the model of the measured signal

$$\tilde{X}(t) = \tilde{X}_0 + \tilde{A}_x \sin \omega t,$$

we get the following ratio for the residuals:

$$I(t) = u(t) + \left[\int_0^t u(\tau)\, d\tau\right] g_1 + \left[t \int_0^t u(\tau) d\tau\right] g_2 + \left[\int_0^t \tau\, u(\tau)\, d\tau\right] g_3 -$$

$$t\,(g_1\tilde{X}_0) - t^2(g_2\tilde{X}_0 + \tfrac{1}{2} g_3\tilde{X}_0) - (g_1\tilde{A}_x)\left[\frac{1}{\omega}(1-\cos\omega t)\right] - (g_2\tilde{A}_x)\left[\frac{1}{\omega} t(1-\cos\omega t)\right] -$$

$$(g_3\tilde{A}_x)\left[\frac{1}{\omega^2}\sin wt - \frac{1}{\omega} t \cos\omega t\right].$$

Introduces the intermediate search for:

$$X_1 = g_1,\; X_2 = g_2,\; X_3 = g_3,\; X_4 = g_1\tilde{X}_0,\; X_5 = g_2\tilde{X}_0 + \frac{1}{2} g_3\tilde{X}_0,$$

$$X_6 = g_1\tilde{A}_x,\; X_7 = g_2\tilde{A}_x,\; X_8 = g_3\tilde{A}_x.$$

Then the residual can be written as:

$$I(t) = u(t) + \left[\int_0^t u(\tau)\,d\tau\right] X_1 + \left[t\int_0^t u(\tau)d\tau\right] X_2 + \left[\int_0^t \tau\, u(\tau)\,d\tau\right] X_3 -$$

$$tX_4 - t^2 X_5 - \left[\frac{1}{\omega}(1-\cos\omega t)\right] X_6 - \left[\frac{1}{\omega} t(1-\cos\omega t)\right] X_7 -$$

$$\left[\frac{1}{\omega^2}\sin\omega t - \frac{1}{\omega} t\cos\omega t\right] X_8.$$

Applying the second method of compiling the basic SLAE, we have:

$$\sum_{i=1}^{8} A_{ki} X_i = B_k, \quad k = 1, \ldots, 8, \; T = T0/7, \; t_k = t_1 + T(k-1),$$

where $B_k = u(t_k)$, $A_{k1} = -\int_0^{t_k} u(\tau)\,d\tau$, $A_{k2} = -t_k \int_0^{t_k} u(\tau)\,d\tau$, $A_{k3} = -\int_0^{t_k} \tau \cdot u(\tau)\,d\tau$, $A_{k4} = t_k$, $A_{k5} = t_k^2$, $A_{k6} = \frac{1}{\omega}(1-\cos\omega t_k)$, $A_{k7} = \frac{1}{\omega} t_k (1-\cos\omega t_k)$, $A_{k8} = \frac{1}{\omega^2}\sin\omega t_k - \frac{1}{\omega} t_k \cos\omega t_k$.

Initial unknown values are determined by the ratios:

$$g_1 = X_1, \; g_2 = X_2, \; g_3 = X_3, \; \tilde{X}_0 = \frac{X_4}{g_1}, \; \tilde{A}_x = \frac{X_6}{g_1}.$$

The expression for X_5, from which the source is unknown $\tilde{X}_0$ can also be defined, as well as expressions for X_7 and X_8, from which the original unknown $\tilde{A}_x$, can be defined, are superfluous. After the solution to the system, these expressions are left without consideration.

Table 6 shows the results of recovery of the third typical signal using the invariance algorithm based on the solution of the above SLAE. Here, the numerical values of the parameters in the source data during modeling are taken:

$$\alpha_1 = 0.1, \quad \alpha_2 = 0.2, \quad X_0 = 100, \quad A_x = 40, \quad \omega = 2\pi F, \quad F = 1 \text{ Hz}.$$

As can be seen from the results given in Table 6, for the measurement system of an arbitrary (unknown) structure, the restoration of the third typical signal with the help of the invariance algorithm is is quite acceptable.

Table 6. **Implementation of the invariance algorithm for the third typical signal at unknown structure of the measuring system**

P	$t_1 \le t \le t_1 + T0$	$T0P$	Indirect Criteria $\rho I_{u\,от}$	Direct Criterion $\rho X_{от}$	Recovered Signal		cond 1 (A)
					$\tilde{X}_0$	$\tilde{A}_x$	
1	[0.04; 0.09]	0.05	0.407	0.499992	100.503347	40.191307	$2.8 \cdot 10^{12}$
2	[0.04; 0.14]	0.1	0.539	$3.35 \cdot 10^{-3}$	100.003377	40.001297	$1.76 \cdot 10^{12}$
3	[0.04; 0.24]	0.2	0.951	$3.54 \cdot 10^{-5}$	100.000035	40.000013	$1.1 \cdot 10^{10}$
4	[0.04; 0.34]	0.3	1.432	$1.65 \cdot 10^{-5}$	100.000016	40.000006	$7.68 \cdot 10^{8}$
5	[0.04; 0.44]	0.4	1.991	$2.62 \cdot 10^{-7}$	100.000000	40.000000	$2.43 \cdot 10^{8}$
6	[0.04; 0.54]	0.5	2.637	$5.88 \cdot 10^{-7}$	100.000000	40.000000	$9.4 \cdot 10^{7}$
7	[0.04; 0.64]	0.6	3.38	$1.84 \cdot 10^{-7}$	100.000000	40.000000	$4.2 \cdot 10^{7}$
8	[0.04; 0.74]	0.7	4.22	$2.02 \cdot 10^{-7}$	99.999999	39.999999	$2.1 \cdot 10^{7}$
9	[0.04; 0.84]	0.8	5.139	$4.99 \cdot 10^{-8}$	100.000000	40.000000	$1.2 \cdot 10^{7}$
10	[0.04; 0.94]	0.9	6.116	$5.82 \cdot 10^{-8}$	100.000000	40.000000	$7.1 \cdot 10^{6}$
11	[0.04; 1.04]	1.0	7.113	$4.92 \cdot 10^{-9}$	100.000000	40.000000	$4.77 \cdot 10^{6}$

Pay attention to the fact that, as expected, the systems on the basis of which invariance algorithms are implemented in this general case, as in the case of measuring systems of the given structures, are unsustainable. However, the entire previous paragraph is devoted to the solution to this problem, so here we will not dwell on this issue.

4.3. SINGLE-CHANNEL PRINCIPLE OF INVARIANCE FOR MEASURING SYSTEMS WITH DISTRIBUTED PARAMETERS

Problem statement

When trying to use the principle of invariance in the dynamics of the MS with distributed parameters, specifically the MS described by differential equations in partial derivatives, we are faced with such a large number of different mathematical models, that attempts to take as a basis the study of a mathematical model, in any way common to the theory of measurement in general, are obviously doomed. Moreover, even if we address a particular area of measurement, the diversity of physical operating principles and design features of the MS results in such a large number of fundamentally different mathematical models that practically useful results can be obtained only by considering mathematical models that are common to at least some MS groups.

In connection with this, we turn to the measurement field in which parametric effects are most pronounced. Therefore, any means of reducing the influence of these effects is extremely desirable. This area of measurement is thermometry, and the most difficult situation is when measuring fluid and gas flow temperatures.

Physical and mathematical models of modern temperature measurement systems of liquids and gases, commonly referred to as heat sensors, are well known [10]. From this area we choose for research a group of heat sensors with the forms of canonical bodies, whose mathematical models of behavior are given in the first chapter.

The presentation of the processes and results of constructing invariance algorithms for heat sensors of these forms are so uniform that it is enough to limit ourselves to a detailed description of the process and the result of constructing an invariance algorithm only for heat sensors with the form of an unbounded plate.

Mathematically, the problem of interaction between the medium, the temperature of which must be measured, and the heat sensor in the form of an unlimited plate, is formulated in the form of:

$$\frac{\partial u(x,t)}{\partial t} = a_0^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \ -R < x < +R, \tag{30}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=-R} + H(t)\left[\theta(t) - u(x,t)\big|_{x=-R}\right] = 0, \tag{31}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=R} + H(t)\left[u(x,t)\big|_{x=R} - \theta(t)\right] = 0, \tag{32}$$

$$u(x,t)\big|_{t=0} = u_0, \tag{33}$$

where $H(t) = \frac{\alpha_k(t)}{\lambda}$; a_0^2, λ – coefficients of temperature conductivity and thermal conductivity of heat sensor material; $\theta(t)$ – measured temperature of medium; $\alpha_k(t)$ – coefficient of convective heat transfer between heat sensor and medium; R - half of the thickness of the plate.

Since according to (31) and (32), the heat exchange between the surfaces of the flat heat sensor and the medium occurs in the same way, taking into account (33), it can be assumed that the temperature distribution inside the heat sensor is symmetrical. With this in mind, the condition (31) can be replaced by the condition of symmetry

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0. \tag{34}$$

The equations for the unknown functions we need, obtained on the basis of the analysis and use of problem (30) - (34) with the parameter $H(t)$, time-variable, the exact solutions of which, as a rule, cannot be obtained, should in the particular case of constant parameter $H(t) \equiv H$ = const lead to solutions for these functions, which are obtained by directly solving the original problem (30) - (34) for a constant value of the parameter $H(t) \equiv H$.

Therefore, we will give the results of the exact solution to the problem (30) - (34) with a constant value of the parameter H, which we will need in the future. As we know, the simplest method of accurately solving this problem is the operational method [10].

Carrying out a Laplace transformation over all the components of the problem (30) – (34) in the assumption of constant parameter $H(t) \equiv H$ = const, write it in forms:

$$\frac{a_0^2 d^2 U(x,s)}{dx^2} - sU(x,s) + \frac{1}{s} u_0 = 0, \tag{35}$$

$$\frac{dU(0,s)}{dx}=0, \tag{36}$$

$$\frac{dU(R,s)}{dx}+H\,[U(R,s)-\theta_L(s)]=0, \tag{37}$$

where $U(x,s)$ – the image of the function (original) $u(x,t)$; $\theta_L(s)$ – the image of the function $\theta(t)$.

The solution to equation (35) has the form

$$U(x,s)-\frac{u_0}{s}=A\ \mathrm{ch}\frac{\sqrt{s}}{a_0}x+B\ \mathrm{sh}\frac{\sqrt{s}}{a_0}x.$$

From the symmetry condition (36) it follows that B = 0, so

$$U(x,s)-\frac{u_0}{s}=A\ \mathrm{ch}\frac{\sqrt{s}}{a_0}x. \tag{38}$$

Substituting (38) in the condition (37), we find the expression for A:

$$A=\frac{\theta_L(s)-\frac{u_0}{s}}{\mathrm{ch}\frac{\sqrt{s}}{a_0}R+\frac{1}{H}\frac{\sqrt{s}}{a_0}\mathrm{sh}\frac{\sqrt{s}}{a_0}R}.$$

The solution itself in types takes the form:

$$U(x,s)-\frac{u_0}{s}=\frac{\left[\theta_L(s)-\frac{u_0}{s}\right]\mathrm{ch}\frac{\sqrt{s}}{a_0}x}{\mathrm{ch}\frac{\sqrt{s}}{a_0}R+\frac{1}{H}\frac{\sqrt{s}}{a_0}\mathrm{sh}\frac{\sqrt{s}}{a_0}R}. \tag{39}$$

When switching to the originals, let us limit ourselves to the consideration of a particular case: namely, let the temperature of the medium be constant, i.e. $\theta(t)\equiv\theta=\text{const}$. Then $\theta_L(s)=\frac{1}{s}\theta$ for the solution in the originals we get [10]:

$$\frac{u(x,t)-u_0}{\theta-u_0}=1-\sum_{n=1}^{\infty}A_n\cos\mu_n\frac{x}{R}\cdot\exp\left(-\mu_n^2\frac{a_0^2t}{R^2}\right), \tag{40}$$

where $A_n=\dfrac{2\sin\mu_n}{\mu_n+\sin\mu_n\cos\mu_n}$, μ_n – the roots of the equation $\mathrm{ctg}\,\mu=\dfrac{\mu}{HR}$.

Further, we need the average surface temperature in the images, i.e. $U(R,s)$ and the mean temperature in the forms - i.e.

$$U_v(s)=\frac{1}{R}\int_0^R U(x,s)\,dx.$$

From the solution in forms (39) we have:

$$U(R,s)-\frac{u_0}{s}=\frac{\left[\theta_L(s)-\frac{u_0}{s}\right]\mathrm{ch}\frac{\sqrt{s}}{a_0}R}{\mathrm{ch}\frac{\sqrt{s}}{a_0}R+\frac{1}{H}\frac{\sqrt{s}}{a_0}\mathrm{sh}\frac{\sqrt{s}}{a_0}R}. \tag{41}$$

$$U_v(s)-\frac{u_0}{s}=\frac{\left[\theta_L(s)-\frac{u_0}{s}\right]}{\mathrm{ch}\frac{\sqrt{s}}{a_0}R+\frac{1}{H}\frac{\sqrt{s}}{a_0}\mathrm{sh}\frac{\sqrt{s}}{a_0}R}\cdot\frac{\mathrm{sh}\frac{\sqrt{s}}{a_0}R}{\frac{\sqrt{s}}{a_0}R}. \tag{42}$$

In the conclusion to the problem statement, let us recall what was said in the first chapter about the considered second model of MS with distributed parameters: namely, a heat sensor with the form of canonical bodies, which are often designed in two versions. In one version, the registration elements are the conductive canonical bodies themselves. In this version, the readings of the heat sensors correspond to their average temperature $u_v(t)$. In another version, canonical bodies made of non-conductive materials are used as so-called substrates, on which the thin layer of conductive films is applied. These are the registration elements. Moreover, due to the negligibly tiny thickness of the film, it is believed that there are no temperature gradients along the film's thickness, and the temperature of the film coincides with the surface temperature of the substrate. Therefore, in the second version, the readings of the heat sensors correspond to the temperature of the film or whatever is the same, the temperature of the surface of the canonical bodies.

Relationship between the readings of measuring systems and the measured signal

Construction of an invariance algorithm is based on the use of an equation that establishes the relationship between the readings of the measuring system and the measured signal.

It was noted above that in the first version of heat sensors of canonical forms, the average volume temperature $u_v(t)$ of the conductive canonical body is recorded. Therefore, for this group of heat sensors it is necessary to find an equation establishing the relationship between the mean volume temperatures $u_v(t)$ of heat sensors, and the measured temperature of medium $\theta(t)$.

In the second version of heat sensors of canonical forms, the temperature of the surface of the canonical body - the temperature of the conductive film is recorded. Therefore, for this group of heat sensors it is necessary to find an equation establishing the relationship between the mean volume temperatures $u_п(t) = u(R,t)$ of the heat sensors, and the measured temperature of medium $\theta(t)$. Finding these two equations is our immediate goal.

Equations of the relationship between the surface temperature and mean volume temperature of canonical bodies

Let us find two equations that establish the specified relationship. Turning to the solution scheme in the forms of the problem (30) – (33), we have

$$U(x,s) - \frac{u_0}{s} = A\,\mathrm{ch}\frac{\sqrt{s}}{a_0}x.$$

Note: this solution takes place regardless of whether the $H(t)$ parameter is a time function or not.

We write this solution for the form of the surface temperature of the canonical body $U_п(s) = U(R,s)$:

$$U_п(s) - \frac{u_0}{s} = A\,\mathrm{ch}\frac{\sqrt{s}}{a_0}R. \tag{43}$$

Next, the averaged solution in forms by body volume:

$$U_v(s) - \frac{u_0}{s} = A\frac{1}{\frac{\sqrt{s}}{a_0}R}\cdot\mathrm{sh}\frac{\sqrt{s}}{a_0}R. \tag{44}$$

Taking the ratio of left and right parts (43) and (44), we get:

$$\frac{U_п(s) - \frac{u_0}{s}}{U_v(s) - \frac{u_0}{s}} = \frac{\frac{\sqrt{s}}{a_0}R\cdot\mathrm{ch}\frac{\sqrt{s}}{a_0}R}{\mathrm{sh}\frac{\sqrt{s}}{a_0}R}. \tag{45}$$

The conclusion, which is important for the future, is that the ratio on the left side (45) was not dependent on the value A, and therefore not dependent on either the boundary condition or in particular, the $H(t)$ parameter.

Expression (45) can be considered as the first equation establishing the relationship between the forms of the temperature on the surface of the body and the volumetric average temperature of the body. It can be used either as a

$$U_п(s) - \frac{u_0}{s} = \frac{\frac{\sqrt{s}}{a_0}R\cdot\mathrm{ch}\frac{\sqrt{s}}{a_0}R}{\mathrm{sh}\frac{\sqrt{s}}{a_0}R}\left[U_v(s) - \frac{u_0}{s}\right], \tag{46}$$

or in the form of

$$U_v(s) - \frac{u_0}{s} = \frac{\mathrm{sh}\frac{\sqrt{s}}{a_0}R}{\frac{\sqrt{s}}{a_0}R\cdot\mathrm{ch}\frac{\sqrt{s}}{a_0}R}\cdot\left[U_п(s) - \frac{u_0}{s}\right]. \tag{47}$$

Now we need to go in the expressions (46), (47) to the originals. This transition is complicated by the fact that fractions in the right parts (46), (47), which are co-factors to $\left[U_v(s) - \frac{u_0}{s}\right]$ and $\left[U_п(s) - \frac{u_0}{s}\right]$ respectively, are not generalized polynomials. In order for these co-factors to be generalized polynomials, we write the ratios (46) and (47) as:

$$\frac{1}{s}\left[U_{\text{п}}(s)-\frac{u_0}{s}\right]= F(s)\ \left[U_v(s)-\frac{u_0}{s}\right] \tag{48}$$

$$\frac{1}{s}\ \left[U_v(s)-\frac{u_0}{s}\right]= \Phi(s)\ \left[U_{\text{п}}(s)-\frac{u_0}{s}\right], \tag{49}$$

where $F(s)=\dfrac{\operatorname{ch}\frac{\sqrt{s}}{a_0}R}{s\cdot\dfrac{\operatorname{sh}\frac{\sqrt{s}}{a_0}R}{\frac{\sqrt{s}}{a_0}R}}$, $\Phi(s)=\dfrac{\dfrac{\operatorname{sh}\frac{\sqrt{s}}{a_0}R}{\frac{\sqrt{s}}{a_0}R}}{s\cdot\operatorname{ch}\frac{\sqrt{s}}{a_0}R}$.

The functions $F(s)$, $\Phi(s)$ represent the ratio of two generalized polynomials and the conditions of the decomposition theorem during the transition to originals, are fulfilled. Assume the functions $f(t)$ and $\varphi(t)$ are the originals of the $F(s)$ and $\Phi(s)$ forms, i.e. $L^{-1}F(s) = f(t)$, $L^{-1}\Phi(s) = \varphi(t)$ where L is the Laplace transform character. Then, going to the originals in (48), (49), we get:

$$\int_0^t[u_{\text{п}}(\tau)-u_0]\,d\tau=\int_0^t f(t-\tau)[u_v(\tau)-u_0]\,d\tau \tag{50}$$

$$\int_0^t[u_v(\tau)-u_0]\,d\tau=\int_0^t \phi(t-\tau)[u_{\text{п}}(\tau)-u_0]\,d\tau\,. \tag{51}$$

Thus, expressions in originals (50), (51) are analogs of expressions (46), (47).

To complete the presentation relating to the first equation establishing the relationship between the mean volume temperature of canonical bodies and the temperature on their surface, it remains to find the values appearing in (50) and (51) originals $f(t)$ and $\varphi(t)$.

We can find the original $\varphi(t)$ by image $\Phi(s)$ using the existing Laplace transformation table [10], according to which

$$L^{-1}\left\{\frac{1}{k\cdot s\cdot\sqrt{s}}th\,(k\sqrt{s})\right\}=1-\sum_{n=1}^{\infty}\frac{8}{(2n-1)^2\pi^2}\cdot e^{-\frac{(2n-1)^2\pi^2\cdot t}{4k^2}}.$$

In our case, $k=\frac{R}{a_0}$, therefore, the expression for the original $\varphi(t)$ has the form

$$\varphi(t)=1-\sum_{n=1}^{\infty}\frac{8}{(2n-1)^2\pi^2}\cdot e^{-\frac{(2n-1)^2\pi^2}{4}\cdot\frac{a_0^2 t}{R^2}}. \tag{52}$$

Incidentally, at t = 0 in the right side the converging series is obtained, the sum of which is equal to one. So when $t = 0$ we have $\varphi(0) = 0$.

Let us refer to the definition of the original $f(t)$. Write the expression for $f(t)$ as:

$$f(t)=L^{-1}F(s)=L^{-1}\left[\frac{F_1(s)}{F_2(s)}\right],$$

where $F_1(s)=\operatorname{ch}\frac{\sqrt{s}}{a_0}R$, $F_2(s)=\dfrac{s\cdot\operatorname{sh}\frac{\sqrt{s}}{a_0}R}{\frac{\sqrt{s}}{a_0}R}$.

According to the expansion theorem we have:

$$L^{-1}\left[\frac{F_1(s)}{F_2(s)}\right]=\sum_{n=1}^{\infty}\frac{F_1(s_n)}{F_2'(s_n)}\cdot e^{s_n t},$$

where s_n – the roots of the polynomial $F_2(s)$.

Find the roots of polynomial $F_2(s)$:

$$F_2(s)=0,\quad s\cdot\operatorname{sh}\frac{\sqrt{s}}{a_0}R=0,$$

where we have a simple root $s = 0$ and countless roots s_n, defined from the ratio:

$$\mathrm{sh}\frac{\sqrt{s}}{a_0}R = 0, \quad \frac{1}{i}\sin\left(i\frac{\sqrt{s}}{a_0}R\right) = 0, \quad i\frac{\sqrt{s}}{a_0}R = n\pi = \mu_n, \quad n = 1, 2, \ldots$$

Now we find the expression for the derivative function $F_2(s)$:

$$F_2'(s) = \frac{1}{2}\left(\frac{1}{\frac{\sqrt{s}}{a_0}R}\cdot \mathrm{sh}\frac{\sqrt{s}}{a_0}R + \mathrm{ch}\frac{\sqrt{s}}{a_0}R\right).$$

Considering that

$$\frac{\sqrt{s_n}}{a_0}R = \frac{1}{i}\mu_n, \quad \mathrm{sh}\,x = \frac{1}{i}\sin(ix), \quad \mathrm{ch}\,x = \cos(ix), \quad \mathrm{sh}(ix) = i\sin x,$$

$$\mathrm{ch}(ix) = \cos x, \quad s_n = -n^2\pi^2\frac{a_0^2}{R^2},$$

we will get

$$\lim_{s\to 0}\frac{F_1(s)}{F_2'(s)} = 1, \qquad \lim_{s\to s_n}\frac{F_1(s)}{F_2'(s)} = 2.$$

Therefore, according to the expansion theorem we have an expression for the original $f(t)$:

$$f(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2 a_0^2 t}{R^2}}. \tag{53}$$

Thus, the first equation establishing the relationship between the mean volume temperature of canonical bodies and the temperature on their surface is represented by the ratio (50) or (51) in which originals $\varphi(t)$ and $f(t)$ are defined by the expressions (52) and (53) respectively.

Now we find the second equation establishing the relationship between the mean volume temperature $u_v(t)$ of the heat sensor, and the temperature on its surface $u_п(t) = u(R,t)$. All members of the thermal conductivity equation are average in volume (30). This will give:

$$\frac{\partial u_v(t)}{\partial t} = \frac{a_0^2}{R}\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0}^{x=R}.$$

Given the boundary conditions (32) and (34), we get

$$\frac{du_v(t)}{dt} + \frac{a_0^2}{R}H(t)\,[u_{ï}(t) - \theta(t)] = 0. \tag{54}$$

This is the second equation sought.

By entering the notation $u_п(t) - u_0 = \upsilon_п(t)$, $u_v(t) - u_0 = \upsilon_v(t)$, $\theta(t) - u_0 = \theta_c(t)$, we write the found relationships (50), (51) and (54) as:

$$\int_0^t \upsilon_{ï}(\tau)\,d\tau = \int_0^t f(t-\tau)\,\upsilon_v(\tau)\,d\tau \tag{55}$$

$$\int_0^t \upsilon_v(\tau)\,d\tau = \int_0^t \varphi(t-\tau)\,\upsilon_{ï}(\tau)\,d\tau \tag{56}$$

$$\frac{d\upsilon_v(t)}{dt} + \frac{a_0^2}{R}H(t)\,[\upsilon_{ï}(t) - \theta_{ñ}(t)] = 0. \tag{57}$$

Now let us return to the above main problem: namely, to find the equations that establish the relationship between the MS and the measured signal. As it is clear from the above, it is necessary to find two independent equations, one of which refers to the heat sensors of canonical forms in the first version and should establish the relationship between the average volume temperature $u_v(t)$ and the measured signal $\theta_c(t)$ - the temperature of the medium. The other equation refers to the heat sensors of canonical forms in the second version, and should establish the relationship between the surface temperature $u_п(t)$ of the heat sensor, and the measured temperature of the medium.

In the future, the heat sensors of canonical forms in the first version will be called heat sensors of the first group, and the heat sensors of canonical forms in the second version will be called heat sensors of the second group. Divide both parts (57) into $a_0^2 H(t)/R$, denote $\alpha(t) = R/a_0^2 H(t)$ and integrate the resulting ratio in the range from 0 to t. This will give:

$$\int_0^t \alpha(\tau) \frac{d\upsilon_v(\tau)}{d\tau}\, d\tau - \int_0^t \theta_c(\tau)\, d\tau + \int_0^t \upsilon_{\text{ï}}(\tau)\, d\tau = 0.$$

Instead of the third term substituting its expression from (55), we get

$$\int_0^t \alpha(\tau) \frac{d\upsilon_v(\tau)}{d\tau}\, d\tau - \int_0^t \theta_c(\tau)\, d\tau = -\int_0^t f(t-\tau)\, \upsilon_v(\tau)\, d\tau. \tag{58}$$

This is the final form of the desired equation, establishing the relationship between the readings of the heat sensors of the first group, and the measured temperature of the medium.

Equation (58) can also be considered as an equation for the volume mean temperature $\upsilon_v(t)$ with the time variable $H(t)$. Getting the exact solution of the specified equation relative to $\upsilon_v(t)$ is the same complex problem as the solution to the original problem (30) – (33) with the variable in time parameter $H(t)$. We should note however, that from the point of view of constructing invariance algorithms, this fact is not of fundamental importance, since the construction of these algorithms does not require the solution to these equations.

Now let us build an equation that establishes the relationship between the readings of the second group of heat sensors, and the measured temperature of the medium. Let us integrate all the components of the equation (57) in the range from 0 to t. This will give:

$$\upsilon_v(t) + \frac{a_0^2}{R}\int_0^t H(\tau)\, \upsilon_{\text{ï}}(\tau) d\tau - \frac{a_0^2}{R}\int_0^t H(\tau)\, \theta_c(\tau)\, d\tau = 0. \tag{59}$$

Using the Leibniz formula, we differentiate both parts of the equation (56). Then, given that $\varphi(0) = 0$, we get

$$\upsilon_v(t) = \int_0^t \frac{d}{dt}[\varphi(t-\tau)]\, \upsilon_{\text{ï}}(\tau) d\tau. \tag{60}$$

Substituting the expression for $\upsilon_v(t)$ of (60) in (59), we have:

$$\int_0^t a(\tau)\upsilon_{\text{ï}}(\tau) d\tau - \int_0^t a(\tau)\theta_c(\tau) d\tau = -\int_0^t \frac{d}{dt}[\varphi(t-\tau)]\, \upsilon_{\text{ï}}(\tau) d\tau = 0. \tag{61}$$

where $a(t) = \dfrac{a_0^2 H(t)}{R}$.

It is possible to show the fairness of the ratio

$$\int_0^t a(\tau)\upsilon_{\text{ï}}(\tau) d\tau - \int_0^t a(\tau)\theta_c(\tau) d\tau = -\int_0^t \frac{d}{dt}[\varphi(t-\tau)]\, \upsilon_{\text{ï}}(\tau) d\tau = 0.$$

Therefore, instead of equation (61), we can use the equivalent equation:

$$\int_0^t \frac{d}{dt}[\varphi(t-\tau)]\, \upsilon_{\text{ï}}(\tau) d\tau = \int_0^t \varphi(t-\tau) \frac{d}{d\tau}[\upsilon_{\text{ï}}(\tau)]\, d\tau. \tag{62}$$

Equation (61) is the desired equation that establishes the relationship between the readings of the second group of heat sensors and the measured temperature of the medium.

Equation (61) can also be considered as an equation for determining the surface temperature of the thermometric body $\upsilon_{п}(t)$ at the time variable $H(t)$. Getting the exact solution to this equation with respect to $\upsilon_{п}(t)$ is also problematic.

Now that both basic equations (58) and (61) are constructed, establishing the relationship between the readings of the heat sensors of both groups and the measured temperature of the medium in the variable parameter $H(t)$, it is necessary to find out whether the solutions to these equations are special cases, results (41) and (42) obtained earlier for canonical bodies at $H(t) = H = \text{const}$.

Let the parameter $H(t)$ be constant in time, i.e. $H(t) \equiv H = \text{const}$. Hence, $\alpha(t) \equiv \alpha = \text{const}$, $a(t) \equiv a = \text{const}$. Then, carrying out the Laplace transformation over the equation for volumetric average temperatures (58), we obtain:

$$\alpha\upsilon_{vL}(s) - \frac{1}{s}\theta_{cL}(s) = -F(s)\, \upsilon_{vL}(s),$$

where $\upsilon_{vL}(s)$, $\theta_{cL}(s)$ – are forms of functions $\upsilon_v(t)$, $\theta_c(t)$ respectively.

From the last ratio we find an expression for the form of the average temperature $\upsilon_{vL}(s)$:

$$\upsilon_{vL}(s) = \frac{\theta_{cL}(s)}{\alpha + F(s)} \cdot \frac{1}{s},$$

or after substituting an expression for $F(s)$ and simplification:

$$U_v(s) - \frac{u_0}{s} = \frac{\theta_L(s) - \frac{u_0}{s}}{\mathrm{ch}\frac{\sqrt{s}}{a_0}R + \frac{1}{H}\frac{\sqrt{s}}{a_0} \cdot \mathrm{sh}\frac{\sqrt{s}}{a_0}R} \cdot \frac{\mathrm{sh}\frac{\sqrt{s}}{a_0}R}{\frac{\sqrt{s}}{a_0}R}. \quad (63)$$

In exactly the same way, we carry out the Laplace transformation over the equation for the temperature of the surface of the heat sensor (61) and find the expression for the image of the temperature of the surface of the heat sensor. Then taking into account the expression for $\Phi(s)$ we get:

$$U_п(s) - \frac{u_0}{s} = \frac{\left[\theta_L(s) - \frac{u_0}{s}\right]\mathrm{ch}\frac{\sqrt{s}}{a_0}R}{\mathrm{ch}\frac{\sqrt{s}}{a_0}R + \frac{1}{H}\frac{\sqrt{s}}{a_0} \cdot \mathrm{sh}\frac{\sqrt{s}}{a_0}R}. \quad (64)$$

Comparing expressions (63) with (42) and (64) with (41), we notice that the compared expressions are the same. Therefore, if in the common equations (58) and (61), fair for the time-variable parameter H(t), put H(t) ≡ H = const, then the resulting expressions (63) and (64) for the images of the average temperature $U_v(s)$ and the temperature on the surface $U_п(s)$ of the heat sensor coincide with the corresponding expressions (42) and (41) obtained from the direct accurate solution of the original boundary problem (30) – (33) at H(t) ≡ H = const. Obviously, this is also true for the originals of medium volume temperatures and the surface temperatures of heat sensors.

Construction of algorithms invariant to parametric effects

In the previous statement, two general equations characterizing the relationship between the readings of measuring systems with distributed parameters and the measured signal are obtained: equation (58) describing the relationship between the mean volume temperature of the heat sensor and the measured temperature of the medium (heat collectors of the first group), and equation (61), describing the relationship between the temperature of the surface of the heat sensor, and the measured temperature of the medium heat sensors of the second group).

When constructing algorithms invariant to parametric effects, we consider each of these groups of measuring systems separately.

Without losing the generality of further results, we can put the initial condition zero, i.e. $u_0 = 0$, then $\upsilon_v(t) = u_v(t)$, $\upsilon_п(t) = u_п(t)$, $\theta_c(t) = \theta(t)$. Henceforth, when constructing invariance algorithms, we will omit the indices «*v*», «п» and denote the output signal $u(t)$, bearing in mind however, that if the algorithm is built for heat sensors of the first group, $u(t)$ means the average temperature of the heat receivers, and if the algorithm is constructed for the second group heat sensors, $u(t)$ means the temperature of the surface of the heat receivers. In addition, the measured temperature of the medium $\theta(t)$, i.e. the input signal, as in the previous paragraphs, will be marked $X(t)$. Taking into account the above, let's proceed with the construction of algorithms invariant to parametric effects.

Invariance algorithm for first group heat sensors

The equation (58) is true for this group of MS's with distributed parameters. After integrating into parts in the first component of the left part and taking into account the above designations, this equation takes the form of:

$$\alpha(t)u(t) - \int_0^t \frac{d\alpha(\tau)}{d\tau}u(\tau)d\tau - \int_0^t X(\tau)d\tau = -\int_0^t f(t-\tau)u(\tau)d\tau. \quad (65)$$

The problem is to construct on the basis of the ratio (65), an algorithm of recovery of the measured signal $X(t)$ from the indications $u(t)$ of the measuring system, which would be invariant to the parametric effects; that is, it would not depend on the value of the variable in time parameter $\alpha(t)$.

Assume, as before, $\tilde{\alpha}(t)$, $\tilde{X}(t)$ are estimates of unknowns $\alpha(t)$, $X(t)$. Then the expression for the residual values will take the form of

$$I_{\text{int }p}(t) = \tilde{\alpha}(t)u(t) - \int_0^t \frac{d\tilde{\alpha}(\tau)}{d\tau} u(\tau)d\tau - \int_0^t \tilde{X}(\tau)d\tau + \int_0^t f(t-\tau)u(\tau)d\tau. \qquad (66)$$

As in clause 3.1, suppose that estimates $\tilde{\alpha}(t)$, $\tilde{X}(t)$ are described by algebraic polynomials:

$$\tilde{\alpha}(t) = \sum_{i=1}^{n1} \tilde{\alpha}_i \cdot t^{i-1}, \quad \tilde{X}(t) = \sum_{j=1}^{n2} \tilde{\beta}_j \cdot t^{j-1},$$

then the residual (66) is written as

$$I_{\text{int }p}(t) = \sum_{i=1}^{n1} \left\{ t^{i-1}u(t) - (i-1)\int_0^t \tau^{i-2}u(\tau)d\tau \right\} \tilde{\alpha}_i - \sum_{j=1}^{n2} \frac{1}{j} t^j \tilde{\beta}_j + \int_0^t f(t-\tau)u(\tau)d\tau. \qquad (67)$$

As elsewhere, instead of the notation of unknowns $\tilde{\alpha}_i$, $\tilde{\beta}_j$ let us introduce a single notation for unknown:

$$\tilde{\alpha}_i = X_i,\ i = 1, \ldots, n1; \quad \tilde{\beta}_j = X_{n1+j},\ j = 1, \ldots, n2.$$

Then the residual takes the form of

$$I_{\text{int }p}(t) = \sum_{i=1}^{n1} \left\{ t^{i-1}u(t) - (i-1)\int_0^t \tau^{i-2}u(\tau)d\tau \right\} X_i -$$

$$\sum_{i=n1+1}^{n1+n2} \frac{1}{i-n1} t^{i-n1} X_i + \int_0^t f(t-\tau)u(\tau)d\tau, \qquad (68)$$

in which $(n1 + n2)$ of unknown X_i appear.

To make the basic SLAE write the residue (68) for $(n1 + n2 + 1)$ points $t_1, \ldots, t_{n1+n2+1}$, requiring equal zero residue at the specified points. Then, starting with the equation written for the point $t = t_2$, subtract from each equation the previous equation, this will give the following SLAE of order $(n1 + n2)$ to determine $(n1 + n2)$ unknown X_i:

$$\sum_{i=1}^{n1} \left\{ t_{k+1}^{i-1}u(t_{k+1}) - t_k^{i-1}u(t_k) - (i-1)\int_{t_k}^{t_{k+1}} \tau^{i-2}u(\tau)d\tau \right\} X_i -$$

$$\sum_{i=n1+1}^{n1+n2} \left\{ \frac{1}{i-n1}\left[t_{k+1}^{i-n1} - t_k^{i-n1} \right] \right\} X_i =$$

$$\int_0^{t_k} f(t_k-\tau)u(\tau)d\tau - \int_0^{t_{k+1}} f(t_{k+1}-\tau)u(\tau)d\tau, \quad k = 1, \ldots, (n1+n2).$$

Assume $T0$ be the length of the recovery interval of the measured signal $X(t)$, then the discrete pitch size T is equal to $T0/(n1 + n2)$ and $t_k = t_1 + + T(k - 1)$, $k = 1, \ldots, (n1 + n2 + 1)$.

Given the above, write down the SLAE in the form of

$$\sum_{i=1}^{n1} \Big\{ (t_1+Tk)^{i-1} \cdot u(t_1+Tk) - [t_1+T(k-1)]^{i-1} \cdot u[t_1+T(k-1)] -$$

$$(i-1)\int_{t_1+T(k-1)}^{t_1+Tk} \tau^{i-2}u(\tau)d\tau \Big\} X_i - \sum_{i=n1+1}^{n1+n2} \left\{ \frac{1}{i-n1}\left[(t_1+Tk)^{i-n1} - [t_1+T(k-1)]^{i-n1} \right] \right\} X_i =$$

$$\int_0^{t_1+T(k-1)} f\,[t_1+T(k-1)-\tau]\,u(\tau)d\tau - \int_0^{t_1+Tk} f\,[t_1+Tk-\tau]\,u(\tau)d\tau. \qquad (69)$$

Finally, by entering the

$$I(k,i) = (i-1)\int_{t_1+T(k-1)}^{t_1+Tk} \tau^{i-2}u(\tau)d\tau,$$

$$\Psi_k = \int_0^{t_1+Tk} f\,[t_1+Tk-\tau]\,u(\tau)d\tau - \int_0^{t_1+T(k-1)} f\,[t_1+T(k-1)-\tau]\,u(\tau)d\tau,$$

write the given SLAE in a standard form:

$$\sum_{i=1}^{n1+n2} A_{ki} X_i = B_k, \quad k = 1, 2, \dots, (n1 + n2), \tag{70}$$

where $B_k = -\Psi_k$.

$$A_{ki} = \begin{cases} (t_1 + Tk)^{i-1} u(t_1 + Tk) - [t_1 + T(k-1)]^{i-1} u[t_1 + T(k-1)] - I(k,i), & i \le n1 \\ -\dfrac{1}{i-n1}\left[(t_1 + Tk)^{i-n1} - [t_1 + T(k-1)]^{i-n1}\right], & i > n1 \end{cases}$$

This is the main SLAE of invariance algorithm when applying single-channel invariance principle to the measured system with distributed parameters - to heat sensors of the first group.

Invariance algorithm for heat sensors of the second group

For this group of MS's with distributed parameters, equation (62) is valid, which can now be written as

$$\int_0^t a(\tau)u(\tau)d\tau - \int_0^t a(\tau)X(\tau)d\tau = -\int_0^t \varphi(t-\tau)\frac{d}{d\tau}u(\tau)d\tau. \tag{71}$$

On the basis of equation (71) it is necessary to construct such an algorithm of recovery of the measured signal $X(t)$ according to the indications $u(t)$ of the measuring system, which would not depend on the value of the variable in time parameter $a(t)$.

Assume $\tilde{a}(t)$, $\tilde{X}(t)$ are estimates of unknown $a(t)$, $X(t)$ respectively, then the expression for the residuals will take the form of

$$I_{\text{int } p}(t) = \int_0^t \tilde{a}(\tau)u(\tau)d\tau - \int_0^t \tilde{a}(\tau)\tilde{X}(\tau)d\tau + \int_0^t \varphi(t-\tau)\frac{d}{dt}u(\tau)d\tau. \tag{72}$$

Next, we use algebraic polynomials for evaluation expressions:

$$\tilde{a}(t) = \sum_{i=1}^{n1} \tilde{\alpha}_i \cdot t^{i-1}, \quad \tilde{X}(t) = \sum_{j=1}^{n2} \tilde{\beta}_j \cdot t^{j-1},$$

where $\tilde{\alpha}_i$, $\tilde{\beta}_j$ –e unknown constants, and $\tilde{\alpha}_i$ now has a different meaning than in the previous case.

Now the residual (72) is written as

$$I_{\text{int } p}(t) = \sum_{i=1}^{n1}\left\{\int_0^t \tau^{i-1}u(\tau)d\tau\right\}\tilde{\alpha}_i - \sum_{i=1}^{n1}\sum_{j=1}^{n2}\left\{\frac{1}{i+j-1}t^{i+j-1}\right\}\tilde{\alpha}_i\tilde{\beta}_j +$$

$$+\int_0^t \varphi(t-\tau)\frac{d}{d\tau}u(\tau)d\tau. \tag{73}$$

When considering the second group of MS's with distributed parameters, the second scheme of constructing the invariance algorithm described earlier, is realized. In this scheme, due to the nonlinear nature of the expression (73) relatively unknown $\tilde{\alpha}_i$, $\tilde{\beta}_j$, it is necessary to introduce a new group of unknown variables, which will reduce the main nonlinear system to some sort of linear system. Introducing a new group of unknowns considered for specific values $n1$, $n2$. We assume $n1 = 4$, $n2 = 4$. Then the residual will take the form of

$$I_{\text{int } p}(t) = \sum_{i=1}^{n1}\left\{\int_0^t \tau^{i-1}u(\tau)d\tau\right\}\tilde{\alpha}_i - t\tilde{\alpha}_1\tilde{\beta}_1 - \frac{1}{2}t^2(\tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1) -$$

$$\frac{1}{3}t^3(\tilde{\alpha}_1\tilde{\beta}_3 + \tilde{\alpha}_2\tilde{\beta}_2 + \tilde{\alpha}_3\tilde{\beta}_1) - \frac{1}{4}t^4(\tilde{\alpha}_1\tilde{\beta}_4 + \tilde{\alpha}_2\tilde{\beta}_3 + \tilde{\alpha}_3\tilde{\beta}_2 + \tilde{\alpha}_4\tilde{\beta}_1) -$$

$$\frac{1}{5}t^5(\tilde{\alpha}_2\tilde{\beta}_4 + \tilde{\alpha}_3\tilde{\beta}_2 + \tilde{\alpha}_4\tilde{\beta}_2) - \frac{1}{6}t^6(\tilde{\alpha}_3\tilde{\beta}_4 + \tilde{\alpha}_4\tilde{\beta}_3) - \frac{1}{7}t^7\tilde{\alpha}_4\tilde{\beta}_4 +$$

$$+\int_0^t \varphi(t-\tau)\frac{d}{d\tau}u(\tau)d\tau.$$

New unknown $\gamma_1, \ldots, \gamma_7$ will be introduced according to the ratios:

$$\gamma_1 = \tilde{\alpha}_1\tilde{\beta}_1,\quad \gamma_2 = \tilde{\alpha}_1\tilde{\beta}_2 + \tilde{\alpha}_2\tilde{\beta}_1,\quad \gamma_3 = \tilde{\alpha}_1\tilde{\beta}_3 + \tilde{\alpha}_2\tilde{\beta}_2 + \tilde{\alpha}_3\tilde{\beta}_1,$$

$$\gamma_4 = \tilde{\alpha}_1\tilde{\beta}_4 + \tilde{\alpha}_2\tilde{\beta}_3 + \tilde{\alpha}_3\tilde{\beta}_2 + \tilde{\alpha}_4\tilde{\beta}_1,\quad \gamma_5 = \tilde{\alpha}_2\tilde{\beta}_4 + \tilde{\alpha}_3\tilde{\beta}_2 + \tilde{\alpha}_4\tilde{\beta}_2,$$

$$\gamma_6 = \tilde{\alpha}_3\tilde{\beta}_4 + \tilde{\alpha}_4\tilde{\beta}_3,\quad \gamma_7 = \tilde{\alpha}_4\tilde{\beta}_4.$$

The number of new unknown is $n3 = n1 + n2 - 1 = 7$. Now the total number of unknowns $\alpha_1, \ldots, \alpha_4, \gamma_1, \ldots, \gamma_7$ is $n1 + n3 = 2n1 + + n2 - 1 = 11$, i.e., by $(n1 - 1) = 3$ is greater than the number initial unknowns $\tilde{\alpha}_i$, $i = 1, \ldots, 4$; $\tilde{\beta}_j$, $j = 1, \ldots, 4$.

After the introduction of a new group of unknowns, the residual value takes the form of

$$I_{\text{int }p}(t) = \sum_{i=1}^{n1}\left\{\int_0^t \tau^{i-1}u(\tau)d\tau\right\}\tilde{\alpha}_i - \sum_{j-1}^{n3}\frac{1}{j}t^j\gamma_j + \int_0^t \varphi(t-\tau)\frac{d}{d\tau}u(\tau)d\tau.$$

Let us introduce a single designation of unknown:

$$\tilde{\alpha}_i = X_i,\ i = 1, \ldots, n1;\quad \gamma_j = X_{j+n1},\ j = 1, \ldots, n3,$$

then the residual value is written as:

$$I_{\text{int }p}(t) = \sum_{i=1}^{n1}\left\{\int_0^t \tau^{i-2}u(\tau)d\tau\right\}X_i - \sum_{i=n1+1}^{n1+n3}\frac{1}{i-n1}t^{i-n1}X_i + \int_0^t \varphi(t-\tau)\frac{d}{dt}u(\tau)d\tau.$$

To compile the basic SLAE, we write this residual expression for $(n1 + n3 + 1) = (2n1 + n2) = 12$ points $t_1, \ldots, t_{12}$ and require equal zero residuals at the specified points. Then, starting with the equation written for the point $t = t_2$, we subtract from each equation the previous equation. These actions will lead to the basic SLAE of the eleventh order with eleven unknowns X_i:

$$\sum_{i=1}^{n1}\left\{\int_{t_k}^{t_{k+1}} \tau^{i-1}u(\tau)d\tau\right\}X_i - \sum_{i=n1+1}^{n1+n3}\frac{1}{i-n1}\left[t_{k+1}^{i-n1} - t_k^{i-n1}\right]X_i =$$

$$-\int_0^{t_{k+1}} \varphi(t_{k+1}-\tau)\frac{d}{d\tau}u(\tau)d\tau + \int_0^{t_k} \varphi(t_k-\tau)\frac{d}{d\tau}u(\tau)d\tau.$$

Assume $T0$ - the length of the recovery interval of the measured signal $X(t)$, then the discrete pitch size T is equal to$T0/(n1 + n3)$ and $t_k = t_1 + + T(k - 1)$, $k = 1, \ldots, (n1 + n3 + 1)$. Now write the received basic SLAE in the standard form:

$$\sum_{i=1}^{n1+n3} A_{ki}X_i = B_k,\quad k = 1, \ldots, (n1 + n3) = 11,$$

where $$B_k = \int_0^{t_k} \varphi(t_k-\tau)\frac{d}{d\tau}u(\tau)d\tau - \int_0^{t_{k+1}} \varphi(t_{k+1}-\tau)\frac{d}{d\tau}u(\tau)d\tau \tag{74}$$

$$A_{ki} = \begin{cases} \int_{t_k}^{t_{k+1}} \tau^{i-1}u(\tau)d\tau, & i \le n1 \\ -\frac{1}{i-n1}(t_{k+1}^{i-n1} - t_k^{i-n1}), & i > n1 \end{cases}$$

The system (74) is the main SLAE of the invariance algorithm when applying the single-channel invariance principle to the considered measuring systems with distributed parameters - to heat sensors of the second group.

After solving this system, the original unknown $\tilde{\alpha}_i$, $i = 1, \ldots, 4$; $\tilde{\beta}_j$, $j = 1, \ldots, 4$ are defined from the expressions:

$$\tilde{\alpha}_i = X_i,\ i = 1, \ldots, 4,\quad \tilde{\beta}_1 = \frac{\gamma_1}{\tilde{\alpha}_1} = \frac{X_5}{X_1},\quad \tilde{\beta}_2 = \frac{X_6 - X_2\tilde{\beta}_1}{X_1},$$

$$\tilde{\beta}_3 = \frac{X_7 - X_2\tilde{\beta}_2 - X_3\tilde{\beta}_1}{X_1}, \quad \tilde{\beta}_4 = \frac{X_8 - X_2\tilde{\beta}_3 - X_3\tilde{\beta}_2 - X_4\tilde{\beta}_1}{X_1}.$$

We will note that the found intermediate unknowns γ_5, γ_6, γ_7, i.e. X_9, X_{10}, X_{11}, were superfluous, so they were not used.

When the values $n1$, $n2$ change, the type of system (74) does not change, only the number of unknown $\tilde{\alpha}_i$, $\tilde{\beta}_j$ changes, as well as the number of intermediate unknown γ_i, and the structure of the relationship between the original and the intermediate unknown.

It is easy to show that assuming the absence of temperature gradients inside the thermometric bodies in question, i.e. $u_v(t) = u_п(t)$, the above basic SLAE for distributed MS parameters are transferred to the main SLAE for the lumped parameters MS.

CHAPTER 5

PARAMETRIC PHENOMENA IN STATISTICAL DYNAMICS OF MEASURING SYSTEMS

Parametric phenomena in the statistical dynamics of measuring systems are considered in detail in [1]. The following results give a general idea of the patterns of manifestation of parametric effects in statistical dynamics of MS's. Later, some of these results are used in the development of the algorithm for excluding the influence of parametric effects on the accuracy of recovery of the statistical characteristics of the measured signal.

Both accurate and approximate methods are used as methods of statistical analysis of MS's, and analysis of dynamic properties is limited to correlation theory. The resulting analytical expressions, which establish the relationship between the statistical characteristics of the MS's response and the measured signal, allow us to estimate the errors in determining the characteristics important for practice - mathematical expectation and variance of the measured signal. Obviously, this estimate is only possible under certain assumptions about the statistical characteristics of the measured signal and MS parameters, as well as about the values of constants included in these characteristics.

Thus, the traditional method for estimating errors in determining the statistical characteristics of a measured signal - a random process - is completely analogous to the traditional method for estimating errors, which is used to determine the characteristics of deterministic signals, namely, it is based on obtaining solutions to direct problems and the necessary assumptions about the properties of the measured signal and MS parameters.

This chapter concludes with the results of model implementation of the principle of invariance as applied to the statistical dynamics of measuring systems.

5.1. ELEMENTS OF THE THEORY OF MARKOV RANDOM PROCESSES

Here is brief information from this theory, following its presentation contained in [6].

In a general case, systems whose parameters are random functions of time can be described by stochastic differential equations of the form:

$$\dot{Y}_i = \sum_{j=1}^{n} [C_{\text{ij}}(U,t) - V_{\text{ij}}(t)]\phi_{\text{ij}}(Y) + \sum_{j=1}^{n} F_{\text{ij}}(U,t)\ X_j(t) + Y_{i0}\delta(t-t_0),$$

$$i = 1, 2, \ldots, n. \tag{1}$$

The following designations are entered here:

$\boldsymbol{Y}$ - vector of phase coordinates, i.e. vector of output signals; $C_{ij}(\boldsymbol{U},t)$, $F_{ij}(\boldsymbol{U},t)$ - deterministic functions of time and vector of random parameters $\boldsymbol{U}$; $V_{ij}(t)$ - parametric noises; in the field of measurements, these are components of parameters of the measuring system, which are random processes, the nature of which causes the appearance of parametric effects; $\varphi_{ij}(\boldsymbol{Y})$ - nonlinear functions of the vector of phase $\boldsymbol{Y}$ coordinates; $X_j(t)$ - the input signal at the j-th input of the system, which for the case of an additive combination of useful signal $S_j(t)$ and interference $N_j(t)$ can be represented as:

$$X_j(t) = S_j(t) + N_j(t).$$

A useful signal $S_j(t)$ can contain both the regular part $S(\boldsymbol{U},t)$, depending on the time and vector of random parameters, and the irregular part $S°(t)$.

Parametric $V_{ij}(t)$ and additive $N_j(t)$ noises, which can generally be correlated, are treated as white Gaussian noises with zero mathematical expectation and correlation functions:

$$k_{\text{ijkl}}^{V}(t,\tau) = M[V_{\text{ij}}(t)V_{\text{kl}}(\tau)] = G_{\text{ijkl}}^{V}(t)\cdot\delta(t-\tau),$$

$$k_{\text{ij}}^{N}(t,\tau) = M[N_i(t)N_j(\tau)] = G_{\text{ij}}^{N}(t)\cdot\delta(t-\tau), \tag{2}$$

where M - the symbol of the operation of mathematical expectation; $G_{ijkl}^{V}(t)$, $G_{ij}^{N}(t)$ - the intensity of parametric and additive noise, which in particular can be constant values; $\delta(t-\tau)$ – δ - a function, which also allowed the initial conditions to be replaced by equivalent input signals.

If the random functions $V_{ij}(t)$, $N_j(t)$ are not white noise, but can be represented as a result of white noise conversion by forming filters described by differential equations, then considering $V_{ij}(t)$, $N_j(t)$ as additional components of the phase coordinate vector of the system and adding the equations of forming filters to equations (1), we obtain an extended system of equations different from system (1) with dimension $m > n$ only.

Special cases of the model (1) are systems of linear stochastic equations of the form:

$$\dot{Y}_i = \sum_{j=1}^{n}[C_{ij}(U,t) + V_{ij}(t)]\,Y_j + \sum_{j=1}^{n} F_{ij}(U,t)\;X_j(t) + Y_{i0}\delta(t - t_0),$$

$$i = 1, 2, \ldots, n. \tag{3}$$

It is known from theory that the phase coordinates of systems described by models (1), (3) are a multidimensional Markov process which, unlike random general processes, is fully characterized by a two-dimensional distribution law: the law of ordinate distribution of the Markov process at any future point in time t depends only on the ordinate value at a given timeτ and does not depend on which ordinate had a random process in the past.

The following notation is introduced:

$f_2(\boldsymbol{y}, \boldsymbol{y}_\tau; t, \tau)$ - two-dimensional probability density of the values of the vectors $\boldsymbol{y}$ and $\boldsymbol{y}_\tau$ at times t and τ respectively, i.e. $\boldsymbol{y} = \boldsymbol{y}(t)$, and $\boldsymbol{y}_\tau = \boldsymbol{y}(\tau)$; $f(\boldsymbol{y}_\tau; \tau)$ – one-dimensional probability density; $p(\boldsymbol{y}; t \mid \boldsymbol{y}_\tau; \tau)$ – conditional probability density or probability density of transition from state $\boldsymbol{y}_\tau = \boldsymbol{y}(\tau)$ to state $\boldsymbol{y} = \boldsymbol{y}(t)$; $\boldsymbol{y}$ has meaning of vector of phase coordinates.

Since the ratio

$$f_2(\boldsymbol{y}, \boldsymbol{y}_\tau; t, \tau) = f(\boldsymbol{y}_\tau; \tau) \cdot p(\boldsymbol{y}; t \mid \boldsymbol{y}_\tau; \tau) \text{ is true,}$$

it is necessary to have equations of one-dimensional probability density and probability density of transition in order to fully describe the Markov process.

Knowing the probability density of the transition and the initial probability distribution, it is possible to integrate the one-dimensional probability density for an arbitrary moment in time:

$$f(\boldsymbol{y}; t) = \int_{-\infty}^{\infty} p(\boldsymbol{y}, t \mid \boldsymbol{y}_\tau; \tau)\,]\; f(\boldsymbol{y}_\tau; \tau) d\boldsymbol{y}_\tau,$$

where $f(\boldsymbol{y}_\tau; \tau)$ – the probability density of the initial value of the vector of phase coordinates.

It is proved that both probabilistic characteristics - one-dimensional probability density $f(\boldsymbol{y}; t)$ and the probability density of transition $p(\boldsymbol{y}; t \mid \boldsymbol{y}_\tau; \tau)$ are determined by the same linear differential equation in partial derivatives - the Fokker-Planck-Kolmogorov equation (FPK), but under different initial conditions.

For one-dimensional probability density of the continuous Markov process, the FPK equation in scalar form is as follows:

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{n}\frac{\partial}{\partial y_i}\left[A_i(\boldsymbol{y},t)f\right] + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial y_i \partial y_j}\left[B_{ij}(\boldsymbol{y},t)f\right], \tag{4}$$

where $A_i(\boldsymbol{y}, t)$, $B_{ij}(\boldsymbol{y}, t)$ - the so-called demolition and diffusion coefficients.

In the particular case of the one-dimensional Markov random process, equation (4) takes the form:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}\left[A(y,t)f(y)\right] + \frac{1}{2}\frac{\partial^2\left[B(y,t)f(y)\right]}{\partial y^2}. \tag{5}$$

If one dimensional probability density $f(\boldsymbol{y}; \tau)$ satisfying the conditions is set as the initial condition for equation (4):

$$f(\boldsymbol{y}; \tau) \geq 0, \qquad \int_{-\infty}^{\infty} f(\boldsymbol{y}; \tau)\, d\boldsymbol{y} = 1,$$

the solution to this equation is one - dimensional probability density $f(\boldsymbol{y}; t)$, satisfying the requirements of non-negativity and normalization. If you take a -function $\delta(\boldsymbol{y} - \boldsymbol{y}_\tau)$, as the initial condition of the equation (4), the solution of the equation is the probability density of transition $p(\boldsymbol{y}; t \mid \boldsymbol{y}_\tau; \tau)$. It follows that the probability density of transition is both a weight function or a Green function of equation (4).

Let the Markov random process be described by a linear system:

$$\dot{Y}_i = \sum_{j=1}^{n} [C_{ij}(t) + V_{ij}(t)]\, Y_j + N_i(t), \quad i = 1, 2, \dots, n, \tag{6}$$

where $C_{ij}(t)$ – matrix of known coefficients; Y_j – components of vector of phase coordinates; $N_i(t)$, $V_{ij}(t)$ – additive and multiplicative Gaussian correlated noises with mathematical expectations $m_i^N(t)$, $m_{ij}^V(t)$ respectively and the correlative functions of the form (2), as well as the mutual correlation functions:

$$k_{ijk}^{VN}(t,\tau) = M\,[\overset{\circ}{V}_{ij}(t)\,\overset{\circ}{N}_k(\tau)] = G_{ijk}^{VN}(t)\cdot\delta(t-\tau),$$

symbol "∘" means the centralization of random functions.

In this case, the drift and diffusion coefficients are determined by the expressions:

$$A_i(y,t) = \sum_{p,q=1}^{n} \left[C_{\mathrm{ip}}(t) + m_{\mathrm{ip}}^V(t) + \frac{1}{2} G_{\mathrm{iqqp}}^V\right] y_p + \frac{1}{2}\sum_{p=1}^{n} G_{\mathrm{ipp}}^{\mathrm{VN}} + m_i^N, \tag{7}$$

$$B_{\mathrm{ij}}(y,t) = \sum_{p,q=1}^{n} G_{\mathrm{ipjq}}^V y_p y_q + \sum_{p=1}^{n} (G_{\mathrm{ipj}}^{\mathrm{VN}} + G_{\mathrm{jpi}}^{\mathrm{VN}})\; y_p + G_{\mathrm{ij}}^N, \tag{8}$$

$$i, j = 1, \dots, n.$$

In the statistical dynamics of measuring systems, we are usually limited to correlative theory. Therefore, let us dwell only on the equations for the first two moments of phase coordinates, which are obtained as a private result from the general theory. From this theory, it is known that the equations for the first and second initial moments of phase coordinates in scalar form are as follows:

$$\dot{m}_i = M\,[A_i(\boldsymbol{Y}, t)], \quad i = 1, 2, \dots, n, \tag{9}$$

$$\dot{\theta}_{ij} = M\,[Y_i A_j + A_i Y_j + B_{ij}]. \tag{10}$$

The first of these equations are integrated at the mathematical expectation of the initial conditions, and the second equations are integrated at the second moments of the initial values of the vector of phase coordinates. Assuming in the formulas (7), (8) phase coordinates random, $\boldsymbol{y} = \boldsymbol{Y}$, we substitute the drift coefficients and diffusion coefficients in the equations (9), (10) and carry out the operation of mathematical expectation by phase coordinates. As a result, the following equations are obtained for the first and second initial moments:

$$\dot{m}_i = \sum_{p,q=1}^{n} \left[C_{ip}(t) + m_{ip}^V(t) + \frac{1}{2} G_{iqqp}^V\right] m_p + \frac{1}{2}\sum_{p=1}^{n} G_{ipp}^{VN} + m_i^N(t), \tag{11}$$

$$\begin{aligned}
\dot{\theta}_{ij} = {} & \sum_{p,q=1}^{n} \left[C_{jp}(t) + \frac{1}{2} G_{iqqp}^V + m_{jp}^V(t)\right] \theta_{ip} + \left(\frac{1}{2}\sum_{p=1}^{n} G_{jpp}^{VN} + m_j^N(t)\right) m_i + \\
& + \sum_{p,q=1}^{n} \left[C_{ip}(t) + \frac{1}{2} G_{iqqp}^V + m_{ip}^V(t)\right] \theta_{pj} + \left(\frac{1}{2}\sum_{p=1}^{n} G_{ipp}^{VN} + m_i^N(t)\right) m_j + \\
& + \sum_{p,q=1}^{n} G_{ipjq}^V\, \theta_{pq} + \sum_{p=1}^{n} \left(G_{ipj}^{VN} + G_{jpi}^{VN}\right) m_p + G_{ij}^N.
\end{aligned} \tag{12}$$

Since the correlation moments of Kij and the second initial moments are connected by the ratio of

$$\theta_{ij} = K_{ij} + m_i m_j,$$

then from (12) the equations for correlation moments are obtained:

$$\begin{aligned}
\dot{K}_{ij} = {} & \sum_{p,q=1}^{n} \left[\left(C_{ip} + \frac{1}{2} G_{iqqp}^V + m_{ip}^V(t)\right) K_{pj} + \left(C_{jp} + \frac{1}{2} G_{jqqp}^V + m_{jp}^V(t)\right) K_{ip} + \right. \\
& \left. + G_{ipjq}^V K_{pq}\right] + \sum_{p=1}^{n} \left(G_{ipj}^{VN} + G_{jpi}^{VN}\right) m_p + G_{ij}^N, \quad i, j = 1, 2, \dots, n.
\end{aligned} \tag{13}$$

The correlation functions of phase coordinates of the system described (6) are determined in the following sequence:

1. Calculation of the matrix of weight functions $\boldsymbol{G}(t, \tau)$ by solving the equation:

$$\frac{d\boldsymbol{G}(t,\tau)}{dt} = \tilde{\boldsymbol{C}}(t)\boldsymbol{G}(t,\tau) + \boldsymbol{I}\delta(t-\tau), \quad \boldsymbol{G}(\tau,\tau) = 0 \tag{14}$$

where matrix $\tilde{\boldsymbol{C}}(t)$ has components:

$$\tilde{C}_{ip} = C_{ip}(t) + \frac{1}{2}\sum_{q=1}^{n} G^{V}_{iqqp}(t) + m^{V}_{ip}(t);$$

$\boldsymbol{I}$ - a single matrix.

Instead of δ-function on the right side (14) it is possible to take single initial conditions for (m - 1) derivative, where m is the order of the differential equation. Therefore, if, for example, m = 1, then, instead of (14), you can solve the equation:

$$\frac{d\boldsymbol{G}(t,\tau)}{dt} = \tilde{\boldsymbol{C}}(t)\boldsymbol{G}(t,\tau), \quad \boldsymbol{G}(\tau,\tau) = \boldsymbol{I}. \tag{15}$$

2. Calculation of the matrix of weight functions $\boldsymbol{G}'(\tau, t)$, conjured with the matrix of weight functions $\boldsymbol{G}(t, \tau)$, by solving the equation:

$$\frac{d\boldsymbol{G}'(\tau,t)}{d\tau} = -\tilde{\boldsymbol{C}}'\boldsymbol{G}'(\tau,t), \quad \boldsymbol{G}'(t,t) = \boldsymbol{I}, \tag{16}$$

where matrix $\tilde{\boldsymbol{C}}'$ – the matrix conjured with matrix $\tilde{\boldsymbol{C}}(t)$.

3. Calculation of the matrix of correlation moments $\boldsymbol{K}(t)$ by solving equations for correlation moments (13).

4. Calculation of matrix of correlation functions $\boldsymbol{k}(t, \tau)$ phase coordinates by formula:

$$\boldsymbol{k}(t,\tau) = 1(t-\tau)\,\boldsymbol{G}(t,\tau)\,\boldsymbol{K}(\tau) + 1(\tau - t)\boldsymbol{K}(t)\boldsymbol{G}'(\tau,t), \tag{17}$$

where $1(t - \tau)$, $1(\tau - t)$ – single functions.

The expression (17) defines the matrix of correlation functions for both $t > \tau$ and t $t < \tau$: if $t > \tau$ only the first term is left in it, and if $t < \tau$ only the second term is left.

A large number of examples of application of the theory of Markov random processes are considered in the paper [6]. Here is the simplest of them, which in [6] is interpreted as a model describing the behavior of some linear tracking system. But the same example can be interpreted as a special case of the first general model (1.I) of a measuring system with concentrated parameters.

So assume n = 1, i.e. the first order MS is investigated, then the original model is:

$$\frac{dY(t)}{dt} + a(t)Y(t) = X(t), \quad Y(0) = 0, \tag{18}$$

in which the initial condition is assumed to be zero, and for ease of writing the index "0" in parameter $a(t)$ is omitted.

Assume the parameter $a(t)$ and the measured signal $X(t)$ are normally distributed stationary and stationary correlated white noises. In the symbols used above, the model (18) can be written as:

$$\dot{Y}(t) = -[C + V(t)]\,Y(t) + X(t), \tag{19}$$

where C - a known constant value, which means a mathematical expectation of the parameter $a(t)$; $V(t)$ - a parametric white noise with zero mathematical expectation and intensity G^V, which means a random component of the parameter $a(t)$, that is, it is a centered random variable $\mathring{a}(t) = (a(t) - C)$ with intensity $G^a = G^V$; $X(t) = m_x + N(t)$, m_x - the mathematical expectation of the input signal, $N(t)$ - an additive white noise with zero mathematical expectation and intensity $G^N = G^x$.

It is assumed that the correlation and mutual correlation functions have the form

$$k_a(t,\tau) = k_V(t,\tau) = G^V\delta\,(t-\tau), k_x(t,\tau) = k_N(t,\tau) = G^N\delta\,(t-\tau),$$

$$k_{\mathrm{ax}}(t,\tau) = k_{\mathrm{VN}}(t,\tau) = G^{\mathrm{VN}}\delta\,(t-\tau).$$

Let us find the equation and the corresponding solution for the mathematical expectation of the output signal. From the general equation (11) for this one-dimensional system at constant intensities we have:

$$\dot{m} + \left(C - \frac{1}{2}G^V\right) m = -\frac{1}{2}G^{\mathrm{VN}} + m_x, m(0) = 0\,. \tag{20}$$

Here $m = M[Y(t)]$ is the mathematical expectation of the output signal.

The solution of equation (20) has the form

$$m(t) = \frac{m_x - 0{,}5G^{VN}}{C - 0{,}5G^V} \cdot \left[1 - e^{-(C-0{,}5G^V)t}\right]. \tag{21}$$

From (21) follows the condition of stability of the system according to the mathematical expectation: $C > 0{,}5G^V$.

If there is no correlation between processes $V(t)$ and $N(t)$ we have:

$$m(t) = \frac{m_x}{C - 0{,}5G^V} \cdot \left[1 - e^{-(C-0{,}5G^V)t}\right]. \tag{21a}$$

For the same case, that is, if there is no correlation between processes $V(t)$ and $N(t)$, the equation for the variance of the output signal is found from the general equation (13):

$$\dot{D} + 2(C - G^V)D = G^N, D(0) = 0. \tag{22}$$

The solution to this equation has the form

$$D(t) = \frac{G^N}{2(C - G^V)} \cdot \left[1 - e^{-2(C-G^V)t}\right]. \tag{23}$$

From (23) the condition of stability of the system for dispersion follows: $C > G^V$.

The expression for the correlation function of the output signal is as follows:

$$k_Y(t,\tau) = 1(t-\tau)\, g(t,\tau)D(\tau) + 1(\tau - t)D(t)g'(\tau,t), \tag{24}$$

where the weight functions $g(t, \tau)$, $g'(\tau, t)$, according to (14) and (16) are as follows:

$$g(t,\tau) = e^{-(C-0{,}5G^V)(t-\tau)}, g'(\tau,t) = e^{-(C-0{,}5G^V)(\tau-t)}.$$

In conclusion, the framework of the theory of Markov processes allows the solving of a much wider and complex class of problems than problems of correlation analysis. However, the complexity of this theory encourages researchers, even in the framework of correlation analysis of systems, to look for alternatives - simpler, approximate methods of research.

5.2. STATISTICAL METHOD OF MOMENTS EQUATIONS

One of the approximate methods is a method that allows us to construct equations for the mathematical expectation and correlation function of the output signal using the initial model of systems, using, in essence, only definitions of the concepts of mathematical expectation, correlation function, mutual correlation function, and properties of the linear operator acting on a random function. As far as we know, this method has been widely used by researchers, at least since the 1960s, although the authors have not identified it as an independent method. In the paper [1] for ease of reference it was called a statistical method of equations of moments; this name will be used in this paper. The approximation of this method is due to the fact that when constructing equations for the mathematical expectation and correlation function of the output signal, to a first approximation, mixed third-order moments are discarded. The study of the range of issues related to the application of this method, including its application in the analysis of MS with distributed parameters, is contained in the work [1]. Here we will limit the application of this method only to the establishment of the relationship between the mathematical expectations of the input and output signals of the MS, that is to find the most important correlation for the statistical analysis of the dynamics of measuring systems.

Let these systems be described by the linear differential equation of n-th order, the coefficients $a_i(t)$ of which, as well as the right part $X(t)$ - the input signal are stationary, and stationary correlated random functions. Let us write this equation in the operator form:

$$L[Y(t); a_i(t)] = X(t), \quad i = 0, 1, \ldots, (n-1); \quad Y(0) = Y'(0) = \ldots\ Y^{(n-1)}(0) = 0. \tag{25}$$

Next, find the mathematical expectation of both parts of this equation, then multiply the right both parts of the equation (25) by a centered random function $\mathring{Y}(t_1) = Y(t_1) - \bar{Y}(t_1)$, where $\bar{Y}(t_1)$ – is the mathematical expectation of the output signal, and find the mathematical expectation of both parts of the expression obtained after multiplying. As a result, taking into account the linearity of the operator, we get equations for the mathematical expectation and correlation function of the output signal of the measuring system.

$$L\,[\bar{Y}(t);\ \bar{a}_i, k_{a_iY}(t,t)] = \bar{X}(t), \tag{26}$$

$$L\,[k_Y(t,t_1);\ \bar{Y}(t), \bar{a}_i, k_{a_iY}(t,t_1)] = k_{XY}(t,t_1), i = 0, 1, \ldots, (n-1), \tag{27}$$

where the following designations are entered: $\overline{Y}(t), \overline{X}(t), \overline{a}_i$ – mathematical expectations of output and input signals, parameters $a_i(t)$ respectively; $k_Y(t, t_1)$ – correlation function of output signal; $k_{a_iY}(t, t_1)$ – mutual correlation function of $a_i(t)$ parameter and output signal $k_{XY}(t, t_1)$ – mutual correlation function of input and output signals.

As we can see, the obtained equations for the mathematical expectation and correlation function of the output signal contain new unknown functions $k_{a_iY}(t, t_1)$ and $k_{XY}(t, t_1)$. The construction of the equations for the new unknown is carried out in the same way as it is done for the correlation function. To get the equations for the new unknown $k_{a_iY}(t, t_1)$, multiply the original equation (25), written by the variable t_1 (instead of t), by centered random functions $\mathring{a}_j(t) = a_j(t) - \overline{a}_j(t)$, $j = 0, 1, \ldots, (n-1)$ and find the mathematical expectation of both parts of the expression obtained after multiplying. To get the equation for the new unknown $k_{XY}(t, t_1)$, multiply the original equation (25), written by the variable t1, with a centered random function t_1, on a centralized random function $\mathring{X}(t) = X(t) - \overline{X}(t)$ and find the mathematical expectation of both parts of the expression obtained after multiplying. As a result of these actions, equations for new unknowns are added to the existing equations (26), (27):

$$L\,[k_{a_jY}(t, t_1);\ \overline{Y}(t_1),\ \overline{a}_i,\ k_{a_ja_i}(t, t_1)] = k_{a_jX}(t, t_1), \quad i, j = 0, 1, \ldots,\ (n-1), \tag{28}$$

$$L\,[k_{XY}(t, t_1);\ \overline{Y}(t_1),\ \overline{a}_i,\ k_{Xa_i}(t, t_1)] = k_X(t, t_1). \tag{29}$$

To determine the mathematical expectation of the output signal, it is necessary to solve a system of $(n + 1)$ equations consisting of equation (26) and equations (28). After solving this system, the functions are known $\overline{Y}(t), k_{a_iY}(t, t_1)$, so as to determine the correlation function of the output signal; it is enough to solve the system consisting of equations (27) and (29). This is the content of the statistical method of moment's equations. The application of this method, generally speaking, does not impose any restrictions either on the distribution laws of random parameters of the system and the measured signal $X(t)$, or on the structure of the correlation functions of parameters and signal.

Since this paragraph is aimed at illustrating the application of the statistical method of moments equations, and the analysis of parametric effects is carried out in the future, consider the situation of measurements in which the random components of the MS and input signal are stationary and permanently correlated white noises; however, no restrictions on the laws of distribution of white noise are imposed.

Enter the following symbols:
$X(t) = \overline{X} + N(t)$, $a_i(t) = \overline{a}_i + \mathring{a}_i(t)$, where $\overline{X}$ и $\overline{a}_i$ – constant values, $\mathring{a}_i(t)$ and $N(t)$ – are stationary and stationary correlated white noises with zero mathematical expectations and intensities of $G^{a}i$, G^N, $G^{a}{}_{i}{}^{N}$, $G^{a}{}_{j}{}^{a}i$, $j = 0, 1, \ldots, (n-1)$.

First general MS model with concentrated parameters

Assume that according to (1.I), the behavior of a non-stationary MS is described by the equation

$$\frac{d^nY(t)}{dt^n} + \sum_{i=0}^{n-1} a_i(t)\frac{d^iY(t)}{dt^i} = X(t), \quad Y(0) = Y'(0) = \ldots = Y^{(n-1)}(0) = 0 \tag{30}$$

According to the general scheme of application of the statistical method of equations of moments, it is necessary to pass from the initial model to the statistical model of investigated MS. For the source model (30), the statistical model (26) - (29) takes the form:

$$\frac{d^n\overline{Y}(t)}{dt^n} + \sum_{i=0}^{n-1}\overline{a}_i\frac{d^i\overline{Y}(t)}{dt^i} = \overline{X} - \sum_{i=0}^{n-1}\frac{d^i}{dt_1^i}k_{a_iY}(t, t_1)\Big|_{t_1=t} \tag{31}$$

$$\frac{d^n}{dt_1^n}k_{a_jY}(t, t_1) + \sum_{i=0}^{n-1}\overline{a}_i\frac{d^i}{dt_1^i}k_{a_jY}(t, t_1) = k_{a_jX}(t, t_1) - \sum_{i=0}^{n-1}\frac{d^i\overline{Y}(t_1)}{dt_1^i}k_{a_ja_i}(t, t_1) \tag{32}$$

$$\frac{d^n}{dt_1^n}k_{XY}(t, t_1) + \sum_{i=0}^{n-1}\overline{a}_i\frac{d^i}{dt_1^i}k_{XY}(t, t_1) = k_X(t, t_1) - \sum_{i=0}^{n-1}\frac{d^i\overline{Y}(t_1)}{dt_1^i}k_{Xa_i}(t, t_1) \tag{33}$$

$$\frac{d^n}{dt^n}k_Y(t, t_1) + \sum_{i=0}^{n-1}\overline{a}_i\frac{d^i}{dt_1^i}k_Y(t, t_1) = k_{XY}(t, t_1) - \sum_{i=0}^{n-1}\frac{d^i\overline{Y}(t)}{dt^i}k_{a_iY}(t, t_1) \tag{34}$$

Here and in the future, if not specified otherwise, we assume $t_1 > t$. Since we will only be interested in the expression for the mathematical expectation of the output signal $Y(t)$, it is sufficient to consider only the system (31) – (32) to achieve this goal. Assume $g(t_1 - \tau)$ – the impulse transient function of the linear stationary system described by the equation

$$\frac{d^n z(t_1)}{dt_1^n} + \sum_{i=0}^{n-1} \bar{a}_i \frac{d^i z(t_1)}{dt_1^i} = X(t_1). \tag{35}$$

Then the solution of equations (32) can be presented as

$$k_{a_j Y}(t, t_1) = \int_0^{t_1} g(t_1 - \eta) \cdot \left[k_{a_j N}(t, \eta) - \sum_{i=0}^{n-1} \frac{d^i \bar{Y}(\eta)}{d\eta^i} k_{a_j a_i}(t, \eta) \right] d\eta. \tag{36}$$

Assume the correlation and mutual correlation functions look like:

$$k_X(t, t_1) = k_N(t, t_1) = G^N \cdot \delta(t_1 - t), \;\; k_{a_i}(t, t_1) = G^{a_i} \cdot \delta(t_1 - t),$$

$$k_{a_i N}(t, t_1) = G^{a_i N} \cdot \delta(t_1 - t), \;\; k_{a_i a_j}(t, t_1) = G^{a_i a_j} \cdot \delta(t_1 - t).$$

Let us refer to the calculation of the sum included in the right part of the expression (31), i.e.:

$$I_1(t) = \sum_{i=0}^{n-1} \frac{d^i}{dt_1^i} k_{a_i Y}(t, t_1)\Big|_{t_1 = t}. \tag{37}$$

By replacing in (36) the index j with i, and the summation index i with k, we have

$$k_{a_i Y}(t, t_1) = \int_0^{t_1} g(t_1 - \eta) \cdot \left[k_{a_i N}(t, \eta) - \sum_{k=0}^{n-1} \frac{d^k \bar{Y}(\eta)}{d\eta^k} k_{a_i a_k}(t, \eta) \right] d\eta. \tag{38}$$

Substitute the expression (38) in (37) and use the Leibniz rule of differentiating the integral with variable limits. If you consider the properties of the δ-function and the fact that the impulse transient function g(t1 – η) satisfies the conditions:

$$\frac{d^i g(t_1 - \eta)}{dt_1^i}\Big|_{\eta = t_1} = \begin{cases} 0 \text{ if } \; i = 0, 1, ..., (n-2) \\ 1 \text{ if } \; i = (n-1), \end{cases}$$

then we can see that all the members of the sum (37), except the latter, are zero. For the last member of the amount, and therefore for the whole amount, we have

$$I_1(t) = \left\{ \int_0^{t_1} g_{t_1}^{(n-1)}(t_1 - \eta) \cdot \left[G^{a_{n-1} N} \cdot \delta(\eta - t) - \sum_{k=0}^{n-1} \frac{d^k \bar{Y}(\eta)}{d\eta^k} G^{a_{n-1} a_k} \cdot \delta(\eta - t) \right] d\eta \right\} \Bigg|_{t_1 = t} =$$

$$= \int_0^{t} g_t^{(n-1)}(t - \eta) \cdot \delta(\eta - t) \left[G^{a_{n-1} N} - \sum_{k=0}^{n-1} \frac{d^k \bar{Y}(\eta)}{d\eta^k} G^{a_{n-1} a_k} \right] d\eta =$$

$$= \frac{1}{2} g_t^{(n-1)}(t - \eta)\Big|_{\eta = t} \cdot \left[G^{a_{n-1} N} - \sum_{k=0}^{n-1} \frac{d^k \bar{Y}(t)}{dt^k} G^{a_{n-1} a_k} \right],$$

or because $g_t^{(n-1)}(t - \eta)|_{\eta = t} = 1$,, going to the previous summation index, we finally get

$$I_1(t) = \frac{1}{2} \left[G^{a_{n-1} N} - \sum_{i=0}^{n-1} \frac{d^i \bar{Y}(t)}{dt^i} G^{a_{n-1} a_i} \right]. \tag{39}$$

The multiplier $\frac{1}{2}$ appears in the expression (39) because here δ-function is different from zero at point $\eta = t$, coinciding with one of the limits of integration. Substituting (39) in the expression for the mathematical expectation (31), we get

$$\frac{d^n \bar{Y}(t)}{dt^n} + \sum_{i=0}^{n-1} \left[\bar{a}_i - \frac{1}{2} G^{a_{n-1} a_i} \right] \frac{d^i \bar{Y}(t)}{dt^i} = \bar{X} - \frac{1}{2} G^{a_{n-1} N}. \tag{40}$$

Assume $g_1(t-\tau)$ – the impulse transient function of the system described by the equation

$$\frac{d^n z(t)}{dt^n}+\sum_{i=0}^{n-1}\left[\bar{a}_i-\frac{1}{2}G^{a_{n-1}a_i}\right]\frac{d^i z(t)}{dt^i}=X(t). \tag{41}$$

Then the solution of equation (40) at zero initial conditions gives the desired analytical expression for the mathematical expectation of the output signal:

$$\overline{Y}(t)=\left[\overline{X}-\frac{1}{2}G^{a_{n-1}N}\right]\cdot\int_0^t g_1(\tau)d\tau. \tag{42}$$

There is an obvious relationship between functions $g(t_1-\tau)$ and $g_1(t-\tau)$: if the function $g(t_1-\tau)$, is known, then to get the function $g_1(t-\tau)$ it is enough to replace in the expression $g(t_1-\tau)$ constant values $\bar{a}_i$ with constant values $\bar{a}_i-\frac{1}{2}G^{a_{n-1}a_i}$, and variable t_1 – with t.

Consider the steady mode of measurement. Using the concept of transfer function $\Phi(s)$ of the system described by equation (41), we have:

$$\left\{\int_0^t g_1(\tau)d\tau\right\}\Bigg|_{t\to\infty}=\lim_{s\to 0}\left\{sL_s\left[\int_0^t g_1(\tau)d\tau\right]\right\}=\lim_{s\to 0}\left(s\frac{1}{s}L_s\left[g_1(t)\right]\right)=$$

$$=\lim_{s\to 0}\Phi(s)=\lim_{s\to 0}\frac{1}{\left(\bar{a}_0-\frac{1}{2}G^{a_{n-1}a_0}\right)+s^n+\sum_{i=1}^{n-1}\left(\bar{a}_i-\frac{1}{2}G^{a_{n-1}a_i}\right)\cdot s^i}=$$

$$=\frac{1}{\bar{a}_0-\frac{1}{2}G^{a_{n-1}a_0}},$$

where L_s – the Laplace transform symbol.

Therefore, for steady modes of measurement we have:

$$\overline{Y}(\infty)=\left(\overline{X}-\frac{1}{2}G^{a_{n-1}N}\right)\frac{1}{\bar{a}_0-\frac{1}{2}G^{a_{n-1}a_0}}. \tag{43}$$

Let us consider a special case of this model of MS. Assume $n=1$, then, instead of the equation (30), we have:

$$\frac{dY(t)}{dt}+a(t)Y(t)=X(t),\quad Y(0)=0, \tag{44}$$

where index "0" for parameter $a(t)$ is omitted.

Since in this case

$$g_1(t-\tau)=\exp\left[-\left(\bar{a}-\frac{1}{2}G^a\right)(t-\tau)\right],$$

then for the mathematical expectation of the MS readings from (42) we immediately get:

$$\overline{Y}(t)=\frac{\overline{X}-\frac{1}{2}G^{aN}}{\bar{a}-\frac{1}{2}G^a}\cdot\left[1-e^{-\left(\bar{a}-\frac{1}{2}G^a\right)t}\right]. \tag{45}$$

Let us compare this result with the one obtained earlier in the study of the non-stationary system of the first order by the method of Markov processes theory, when the initial equation was taken in the form of:

$$\dot{Y}(t)=-[C+V(t)]Y(t)+X(t),\quad X(t)=m_x+N(t). \tag{46}$$

By comparing (44) and (46) we have $a(t)=C+V(t)$, $\bar{a}=C$, $G^a=G^V$, $G^{aN}=G^{VN}$, $m_x=\overline{X}$. Hence, the equation (40) for $n=1$ and the solution (45) can be represented as:

$$\frac{d\overline{Y}(t)}{dt}+\left[C-\frac{1}{2}G^V\right]\overline{Y}(t)=m_x-\frac{1}{2}G^{VN},$$

$$\overline{Y}(t)=\frac{m_x-\frac{1}{2}G^{VN}}{C-\frac{1}{2}G^V}\cdot\left[1-e^{-\left(C-\frac{1}{2}G^V\right)t}\right].$$

Comparing these equations and solution with the obtained earlier method of Markov processes theory, corresponding with equation (20) and solution (21), we note that in this particular case the results for the mathematical expectation are the same.

Note that the use of the total ratio (42) for the mathematical expectation of the output signal is very simple in the case of higher order measuring systems. Similarly, the method described is applied to the second concentrated MS model and to the distributed MS.

5.3. PARAMETRIC EFFECTS IN DYNAMICS OF MEASURING SYSTEMS

Now, unlike the previously accepted conditions, let us assume that the MS's parameters and the input signal are stationary, and stationary correlated normally distributed random processes, which by their statistical properties may differ from white Gaussian noises.

In the analysis below, the main attention will be focused on establishing the sources of parametric effects and elucidating the general laws of their manifestation, i.e., priority will now be given to the qualitative, physical side of the process of reconstructing the measured signal. As in the case of the deterministic analysis carried out in Section 1.4, a study of these questions can be carried out in relation to the simplest representatives of measuring systems, since even in this case it is possible to find out the most important features characteristic of non-stationary MSs. It is clear that the study of the general patterns of manifestation of parametric effects is preferably carried out on the basis of accurate analysis results. These results can be obtained, for example, by the method of statistical analysis using the concept of characteristic function [12].

Parametric effects in the dynamics of MSs with lumped parameters

Below we will sequentially consider particular cases, namely, n = 1, of both general models (1.I), (1.II) of measuring systems with lumped parameters.

First MS model

So, we consider measuring systems, the behavior of which is described by the equation

$$\frac{dY(t)}{dt} + a(t)Y(t) = X(t), \quad Y(0) = 0, \tag{47}$$

where $a(t)$, $X(t)$ – stationary and stationary correlated normally distributed random processes.

It is necessary to find an expression for the mathematical expectation $\overline{Y}(t)$ and the correlation function $k_Y(t, t_1)$ of the output signal, assuming the arbitrariness of the structures of correlation and mutual correlation functions $k_a(t, t_1)$, $k_X(t, t_1)$, $k_{aX}(t, t_1)$.

The solution of equation (47) has the form

$$Y(t) = \int_0^t X(\tau) \cdot e^{-\int_\tau^t a(\eta)d\eta} d\tau. \tag{48}$$

The mathematical expectation of the output signal $\overline{Y}(t)$ is described by the expression:

$$\overline{Y}(t) = \int_0^t M\,[X(\tau) \cdot e^{-\mu(\tau)}]d\tau, \quad \mu(\tau) = \int_\tau^t a(\eta)\,d\eta, \tag{49}$$

where M - the symbol of the mathematical expectation operation.

Let's introduce a system of random values with components $z_1 = X(\tau)$, $z_2 = \mu(\tau)$. Then the expression (49) can be written as

$$\overline{Y}(t) = \frac{1}{i}\int_0^t \left.\frac{\partial E(\lambda_1, \lambda_2)}{\partial \lambda_1}\right|_{\lambda_1 = 0,\, \lambda_2 = i} d\tau, \tag{50}$$

where $E(\lambda_1, \lambda_2)$ – a characteristic function of the system of random variables in question; i - an imaginary unit.

Due to the normality of the system of random values, the characteristic function has the form

$$E(\lambda_1, \lambda_2) = \exp\left[-\frac{1}{2}\sum_{j,p=1}^{2} k_{jp}\lambda_j\lambda_p + i\sum_{j=1}^{2}\bar{z}_j\lambda_j\right], \tag{51}$$

where $\bar{z}_j$ - the mathematical expectations, and k_{jp} - the correlation moments of the component of a random vector.

Since

$$\left.\frac{\partial E(\lambda_1,\lambda_2)}{\partial \lambda_1}\right|_{\lambda_1=0,\ \lambda_2=i} = \left[-ik_{12}+i\bar{z}_1\right]\exp\left[-\bar{z}_2+\frac{1}{2}k_{22}\right],$$

then the expression for mathematical expectation takes the form of

$$\overline{Y}(t)=\int_0^t [\bar{z}_1-k_{12}]\ \exp\left[-\bar{z}_2+\frac{1}{2}k_{22}\right]d\tau. \tag{52}$$

The values included in this expression have the following meaning:

$$\bar{z}_1 = M\,[X(\tau)]=\overline{X},\quad \bar{z}_2=M\left[\int_\tau^t a(\eta)d\eta\right]=\bar{a}(t-\tau)$$

$$k_{12}=M\Big([X(\tau)-\overline{X}]\ [\mu(\tau)-\bar{\mu}(\tau)]\Big)=\int_\tau^t k_{Xa}(\tau,\eta)d\eta=\int_0^{t-\tau}k_{Xa}(\varepsilon)d\varepsilon$$

$$k_{22}=M\big([\mu(\tau)-\bar{\mu}(\tau)][\mu(\tau)-\bar{\mu}(\tau)]\big)=\int_\tau^t\int_\tau^t k_a(\eta_1,\eta_2)d\eta_1 d\eta_2$$

The following formula for the double integral of the correlation function of a stationary random process is valid:

$$\int_\tau^t\int_{\tau_1}^{t_1}k(\eta_2-\eta_1)d\eta_1 d\eta_2=\int_\tau^t d\eta_1\int_{\tau_1}^{t_1}k(\eta_2-\eta_1)d\eta_2=$$

$$=(t-t_1)\int_0^{t_1-t}k(\eta)d\eta+(\tau-\tau_1)\int_0^{\tau_1-\tau}k(\eta)d\eta+(t_1-\tau)\int_0^{t_1-\tau}k(\eta)d\eta+$$

$$+(\tau_1-t)\int_0^{\tau_1-t}k(\eta)d\eta-\int_{\tau_1-\tau}^{\tau_1-t}\eta\cdot k(\eta)d\eta-\int_{t_1-t}^{t_1-\tau}\eta\cdot k(\eta)d\eta. \tag{53}$$

Applying this formula to the expression for k_{22}, we have:

$$k_{22}=2\left[(t-\tau)\int_0^{t-\tau}k_a(\eta)d\eta-\int_0^{t-\tau}\eta\cdot k_a(\eta)d\eta\right].$$

Therefore, for the mathematical expectation of the output signal we will finally get

$$\overline{Y}(t)=\int_0^t\left[\overline{X}-\int_0^{t-\tau}k_{Xa}(\varepsilon)d\varepsilon\right]\exp\left[-\bar{a}(t-\tau)+(t-\tau)\int_0^{t-\tau}k_a(\eta)d\eta-\int_0^{t-\tau}\eta\cdot k_a(\eta)d\eta\right]d\tau. \tag{54}$$

It is easy to see that the general relation (54) implies, as a particular case, the result (21) related to white noise and given earlier in the study of model (47) using the theory of Markov processes. Let us refer to the qualitative conclusions arising from the overall ratio (54).

Assume MS be stationary systems, i.e. $a(t)=\bar{a}=a$, $k_{Xa}(\varepsilon)=0$, $k_a(\eta)=0$, then from (54) we have

$$\overline{Y}(t)=\frac{1}{a}\overline{X}(1-e^{-at}), \tag{55}$$

which in a steady state gives $\bar{Y}(\infty)=\frac{1}{a}\bar{X}$. In this case, the factor $\frac{1}{a}$ can be considered as a coefficient of coordination of dimensions.

Further, if parameter $a(t)$ is a stationary random process not correlated with the measured signal $X(t)$, then the relationship between the mathematical expectations of MS $\overline{Y}(t)$ and the measured signal $\overline{X}(t)$ will be determined by the ratio:

$$\Delta_1(t)=\overline{X}(t)-\overline{Y}(t)=\overline{X}\left\{1-\int_0^t\exp\left[-\bar{a}(t-\tau)+(t-\tau)\int_0^{t-\tau}k_a(\eta)d\eta-\int_0^{t-\tau}\eta\cdot k_a(\eta)d\eta\right]d\tau\right\}. \tag{56}$$

It follows from the expression (56) that in this case there is an offset between the mathematical expectations $\overline{Y}(t)$ and $\overline{X}(t)$ in the steady mode of measurement, and the absolute value of the specified offset depends on the value of the mathematical expectation of the measured signal.

Finally, if the parameter $a(t)$ and the measured signal $X(t)$ are correlated, the component is added to offset $\Delta_1(t)$:

$$\Delta_2=\int_0^t\left\{\left[\int_0^{t-\tau}k_{Xa}(\varepsilon)d\varepsilon\right]\exp\left[-\overline{a}(t-\tau)+(t-\tau)\int_0^{t-\tau}k_a(\eta)d\eta-\int_0^{t-\tau}\eta\cdot k_a(\eta)d\eta\right]\right\}d\tau. \tag{57}$$

Now, we will find the expression for the correlation function of the output signal, for which we will first find the expression for the mixed moment of the second order. Given the type of solution (48), we have:

$$M\left[Y(t)Y(t_1)\right]=\int_0^t\int_0^{t_1}M\left[X(\tau)\cdot e^{-\mu(\tau)}\cdot X(\tau_1)\cdot e^{-\mu(\tau_1)}\right]d\tau\,d\tau_1, \tag{58}$$

$$\mu(\tau)=\int_\tau^t a(\eta_1)d\eta_1,\quad \mu(\tau_1)=\int_{\tau_1}^{t_1}a(\eta_2)d\eta_2,$$

Introduces the normal system of random variables with components:

$$z_1=X(\tau),\quad z_2=\mu(\tau),\quad z_3=X(\tau_1),\quad z_4=\mu(\tau_1).$$

Using the characteristic function $E(\lambda_1, \dots, \lambda_4)$ of the system we have:

$$M\left[Y(t)Y(t_1)\right]=\frac{1}{i^2}\int_0^t\int_0^{t_1}\frac{\partial E(\lambda_1,\dots,\lambda_4)}{\partial\lambda_1\,\partial\lambda_3}\bigg|_{\substack{\lambda_1=\lambda_3=0\\ \lambda_2=\lambda_4=i}}d\tau\,d\tau_1. \tag{59}$$

Since the characteristic function for the normal system now looks like

$$E(\lambda_1,\dots,\lambda_4)=\exp\left[-\frac{1}{2}\sum_{j,p=1}^{4}k_{jp}\lambda_j\lambda_p+i\sum_{j=1}^{4}\overline{z}_j\lambda_j\right],$$

then

$$\frac{\partial E(\lambda_1,\dots,\lambda_4)}{\partial\lambda_1\,\partial\lambda_3}\bigg|_{\substack{\lambda_1=\lambda_3=0\\ \lambda_2=\lambda_4=i}}=\left[-k_{13}-(k_{12}+k_{14}-\overline{z}_1)\,(k_{23}+k_{34}-\overline{z}_3)\right]\times$$

$$\times\exp\left[-(\overline{z}_2+\overline{z}_4)+\frac{1}{2}(k_{22}+k_{44})+k_{24}\right].$$

Substituting this expression in (59), we get the ratio for the mixed moment of the second order:

$$M\left[Y(t)Y(t_1)\right]=\int_0^t\int_0^{t_1}\left[k_{13}+(k_{12}+k_{14}-\overline{z}_1)\,(k_{23}+k_{34}-\overline{z}_3)\right]\times$$

$$\times\exp\left[-(\overline{z}_2+\overline{z}_4)+\frac{1}{2}(k_{22}+k_{44})+k_{24}\right]d\tau d\tau_1. \tag{60}$$

Correlation function of the output signal will be determined by the expression:

$$k_Y(t,t_1)=M\left[Y(t)Y(t_1)\right]-\overline{Y}(t)\overline{Y}(t_1), \tag{61}$$

in which the terms are already found.

Let's reveal the meaning of the values included in the general expression (60):

$$\overline{z}_1=\overline{X},\quad \overline{z}_2=\overline{a}(t-\tau),\quad \overline{z}_3=\overline{X},\quad \overline{z}_4=\overline{a}(t_1-\tau_1)$$

$$k_{12}=\int_\tau^t k_{aX}(\tau-\eta_1)d\eta_1,\quad k_{13}=k_X(\tau_1-\tau),\quad k_{14}=\int_{\tau_1}^{t_1}k_{aX}(\tau-\eta_2)d\eta_2,$$

$$k_{23}=\int_\tau^t k_{aX}(\tau_1-\eta_1)d\eta_1,\quad k_{34}=\int_{\tau_1}^{t_1}k_{aX}(\tau_1-\eta_2)d\eta_2,\quad k_{22}=\int_\tau^t\int_\tau^t k_a(\eta_2-\eta_1)d\eta_1 d\eta_2,$$

$$k_{24}=\int_\tau^t\int_{\tau_1}^{t_1}k_a(\eta_2-\eta_1)d\eta_1 d\eta_2,\quad k_{44}=\int_{\tau_1}^{t_1}\int_{\tau_1}^{t_1}k_a(\eta_2-\eta_1)d\eta_1 d\eta_2.$$

To calculate double integrals, you can use the formula (53) again.

Note that if the parameter $a(t)$ and the measured signal $X(t)$ are not correlated, then the results (23), (24) related to white noise are derived from the general ratios (60), (61) as specific cases given earlier in the study of the model (47) by the method of Markov processes theory.

Let us refer to qualitative conclusions related to different measurement conditions. Since the above conclusions are made regarding the mathematical expectation of the output signal, here, we will carry out this analysis only in relation to the mixed moment of the second order.

Turning to the total ratio (60), consider the same three situations as in the analysis of the mathematical expectation.

Assume MS be stationary systems, then $a(t) = \bar{a} = a$; $k_a(\eta) = k_{aX}(\eta) = 0 \Rightarrow k_{12} = k_{14} = k_{23} = k_{34} = k_{22} = k_{24} = k_{44} \equiv 0$. In this case, for the mixed moment of the second order we have:

$$\left(M\left[Y(t)Y(t_1)\right]\right)_1 = \int_0^t \int_0^{t_1} \left[\overline{X}^2 + k_X(\tau_1 - \tau)\right] \exp\left[-a(t + t_1 - \tau - \tau_1)\right] d\tau d\tau_1. \tag{62}$$

Assume now the parameter $a(t)$ is a stationary random process, not correlated with the measured signal $X(t)$, then $k_{12} = k_{14} = k_{23} = k_{34} \equiv 0$ and from the ratio (60) we have

$$\left(M\left[Y(t)Y(t_1)\right]\right)_2 = \int_0^t \int_0^{t_1} \left[\overline{X}^2 + k_X(\tau_1 - \tau)\right] \exp\left[-(\bar{z}_2 + \bar{z}_4) + \right.$$

$$\left. + \frac{1}{2}(k_{22} + k_{44}) + k_{24}\right] d\tau d\tau_1, \tag{63}$$

where $\bar{z}_2$, $\bar{z}_4$, k_{22}, k_{24}, k_{44} are defined by the above expressions.

Considering jointly the ratio (62), (63), it is possible to assess the impact on the quality of measurements of the parametric effect, due precisely to the statistical nature of the parameter $a(t)$, but not its correlation with the measured signal. The specified evaluation is as follows:

$$\delta = \left(M\left[Y(t)Y(t_1)\right]\right)_2 - \left(M\left[Y(t)Y(t_1)\right]\right)_1.$$

Finally, when the parameter $a(t)$ is correlated and the measured signal $X(t)$, the second order mixed moment for the output signal is determined by the overall ratio (60), which indicates that in this case the parametric effect is added to that which is already mentioned above, due to the correlation of the parameter $a(t)$ and the measured signal $X(t)$. To estimate this parametric effect, it is enough to calculate the difference between the right parts of the ratios (60) and (63).

Second MS model

We consider measuring systems, the behavior of which is described by the equation

$$\frac{dY(t)}{dt} + a(t)Y(t) = a(t)X(t), \quad Y(0) = 0. \tag{64}$$

The solution of equation (64) has the form

$$\overline{Y}(t) = \int_0^t a(\tau)X(\tau)\exp[-\mu(\tau)]d\tau, \quad \mu(\tau) = \int_\tau^t a(\eta)\,d\eta. \tag{65}$$

Find the mathematical expectation of both parts (65), this will give:

$$\overline{Y}(t) = \int_0^t M\left\{a(\tau)\,X(\tau)\exp[-\mu(\tau)]\right\}d\tau. \tag{66}$$

Introduces the system of normally distributed random variables with components:

$$z_1 = a(\tau), \; z_2 = X(\tau), \; z_3 = \mu(\tau).$$

Assume $E(\lambda_1, \lambda_2, \lambda_3)$ – a characteristic function of this system. Then

$$\overline{Y}(t) = \frac{1}{i^2}\int_0^t \left.\frac{\partial^2 E(\lambda_1, \lambda_2, \lambda_3)}{\partial\lambda_1\,\partial\lambda_2}\right|_{\substack{\lambda_1 = \lambda_2 = 0 \\ \lambda_3 = i}} d\tau, \tag{67}$$

where

$$E(\lambda_1, \lambda_2, \lambda_3) = \exp\left[-\frac{1}{2}\sum_{j,p=1}^{3} k_{jp}\lambda_j\lambda_p + i\sum_{j=1}^{3}\bar{z}_j\lambda_j\right].$$

From the expression $E(\lambda_1, \lambda_2, \lambda_3)$ we have:

$$\left.\frac{\partial^2 E(\lambda_1, \lambda_2, \lambda_3)}{\partial\lambda_1\,\partial\lambda_2}\right|_{\substack{\lambda_1=\lambda_2=0\\ \lambda_3=i}} =$$

$$-\left[k_{12} + k_{13}k_{23} - k_{13}\bar{z}_2 - k_{23}\bar{z}_1 + \bar{z}_1\bar{z}_2\right]\exp\left[-\bar{z}_3 + \frac{1}{2}k_{33}\right].$$

Substituting this expression in (67), we get

$$\overline{Y}(t) = \int_0^t e^{(-\bar{z}_3 + \frac{1}{2}k_{33})}\left[k_{12} + k_{13}k_{23} - k_{13}\bar{z}_2 - k_{23}\bar{z}_1 + \bar{z}_1\bar{z}_2\right]d\tau. \quad (68)$$

Let us reveal the meaning of the values included in the expression (68):

$$\bar{z}_1 = \bar{a}, \quad \bar{z}_2 = \overline{X}, \quad \bar{z}_3 = M\left[\int_\tau^t a(\eta)d\eta\right] = \bar{a}(t-\tau);$$

$$k_{12} = k_{aX}(0), \; k_{13} = \int_\tau^t k_a(\tau,\eta)d\eta, \; k_{23} = \int_\tau^t k_{Xa}(\tau,\eta)d\eta, \; k_{33} = \int_\tau^t\int_\tau^t k_a(\eta_1,\eta_2)d\eta_1 d\eta_2$$

Applying the formula (53) to the calculation of k_{33}, we get:

$$k_{33} = 2\left[(t-\tau)\int_0^{t-\tau} k_a(\eta)d\eta - \int_0^{t-\tau}\eta k_a(\eta)d\eta\right],$$

Next, take into account that:

$$k_{13} = \int_\tau^t k_a(\eta-\tau)d\eta = \int_0^{t-\tau} k_a(\eta)d\eta, \quad k_{23} = \int_\tau^t k_{Xa}(\eta-\tau)d\eta = \int_0^{t-\tau} k_{Xa}(\eta)d\eta.$$

Substituting expressions for k_{12}, k_{13}, k_{23}, k_{33} in (68) and making a replacement $t - \tau = \varepsilon$, we get the final expression for the mathematical expectation of the measurement system:

$$\overline{X} - \overline{Y}(t) = \overline{X}\cdot\exp\left[-\bar{a}t + t\int_0^t k_a(\eta)d\eta - \int_0^t \eta k_a(\eta)d\eta\right] +$$

$$+\int_0^t\left[k_{aX}(0) + \left(\int_0^{t-\tau} k_a(\eta)d\eta\right)\cdot\left(\int_0^{t-\tau} k_{Xa}(\eta)d\eta\right) - \bar{a}\int_0^{t-\tau} k_{Xa}(\eta)d\eta\right]\times$$

$$\times\exp\left[-\bar{a}(t-\tau) + (t-\tau)\int_0^{t-\tau} k_a(\eta)d\eta - \int_0^{t-\tau}\eta k_a(\eta)d\eta\right]d\tau. \quad (69)$$

Consider the same three measurement situations as in previous cases. Assume MS to be stationary systems, then a$a(t) = \bar{a} = a$; $k_a(\eta) = k_{aX}(\eta) \equiv 0$, and from the total ratio (69) we get

$$\overline{Y}(t) - \overline{X} = -\overline{X}\exp[-at], \quad \overline{Y}(\infty) = \overline{X}. \quad (70)$$

That is, in steady mode for the stationary MS of this model, the mathematical expectations of the MS and the measured signal coincide.

If the parameter $a(t)$ is a stationary random process that is not correlated with the measured signal $X(t)$, then from (69) we have

$$\Delta_1(t) = \overline{X}(t) - \overline{Y}(t) = \overline{X}\exp\left[-\bar{a}t + t\int_0^t k_a(\eta)d\eta - \int_0^t \eta k_a(\eta)d\eta\right], \quad \Delta_1(\infty) = 0. \quad (71)$$

It follows from the expression (71) that for the MS described in the second model (64), the parametric effect is due only to the statistical nature of the parameter $a(t)$, but not to its correlation with the measured signal, which manifests itself only in the transitional stage of measurement; in the steady state, the effect of this disappears. This is the main difference between the behaviour of the MS described in the second model (64) and the behaviour of the MS described in the first model (47).

Finally, if the parameter $a(t)$ and the measured signal $X(t)$ are correlated, a new component appears in the MS readings due precisely to the correlations of processes $a(t)$ and $X(t)$. The effect of this parametric effect is estimated by the second component of the right part of the total ratio (69). The use of the concept of a characteristic function also makes it possible to elucidate for the second MS model the patterns of influence of parametric effects on the dispersion of MS readings [1].

Parametric effects in the dynamics of MS with distributed parameters

In conclusion, consider the first of the models of measuring systems with distributed parameters presented in chapter 1, described by equations in partial derivatives of parabolic type.

This model has the form:

$$\frac{\partial u(x,t)}{\partial t} = a_0^2 \frac{\partial^2 u(x,t)}{\partial x^2} + m(t)\cdot[\theta(t) - u(x,t)], \quad 0 < x < l, \quad t > 0, \tag{72}$$

$$\frac{\partial u(0,t)}{\partial x} = 0,\ u(l,t) = u_{\text{ст}} = \text{const},\ u(x,0) = u_0 = \text{const},\ m(t) = \frac{\beta}{c\gamma}\cdot\alpha_k(t).$$

The physical meaning of all the symbols used here is given in chapter 1, but now we will consider the parameter $m(t)$ and the measured signal $\theta(t)$ stationary and stationary correlated normally distributed random processes.

As before for this model, in order to simplify the analysis without limiting the commonalities, let us assume $u_0 = u_{\text{ст}}$. After making a replacement $\upsilon(x, t) = u(x, t) - u_0$, write the model (72) as:

$$\frac{\partial \upsilon(x,t)}{\partial t} = a_0^2 \frac{\partial^2 \upsilon(x,t)}{\partial x^2} + m(t)\cdot[V_0(t) - \upsilon(x,t)], \quad 0 < x < l, \quad t > 0, \tag{73}$$

$$\frac{\partial \upsilon(0,t)}{\partial x} = 0, \quad \upsilon(l,t) = 0,\ \upsilon(x,0) = 0, \quad V_0(t) = \theta(t) - u_0.$$

In the first chapter it was shown that if the solution of boundary value problem (73) is searched in the form of

$$\upsilon(x,t) = \sum_{i=1}^{n} a_i(t)\,\varphi_i(x), \tag{74}$$

where $\varphi_i(x)$ – known coordinate functions

$$\varphi_i(x) = \sin\,(2i-1)\frac{\pi}{2l}(x+l),$$

then the unknown functions $a_i(t)$ must be determined from the differential equations:

$$\frac{da_i(t)}{dt} + [\nu_i + m(t)]\,a_i(t) = \frac{4}{\pi}\frac{1}{2i-1} m(t) V_0(t), \quad \nu_i = a_0^2\left[\frac{(2i-1)\pi}{2l}\right]^2, \tag{75}$$

the solutions of which are:

$$a_i(t) = \frac{4}{\pi}\frac{1}{2i-1}\int_0^t m(\tau)V_0(\tau)\exp\left[-\nu_i(t-\tau) - \int_\tau^t m(\eta)d\eta\right]d\tau. \tag{76}$$

We will establish the relationship between the mathematical expectations of the local MS response and the measured signal. By introducing an infinite set of functions $\{\varphi_i(x)\}$, $\{a_i(t)\}$, we get the solution of the boundary value problem:

$$\upsilon(x,t) = \frac{4}{\pi}\sum_{i=1}^{\infty}\frac{1}{2i-1}\sin\left[(2i-1)\frac{\pi}{2l}(x+l)\right]\cdot\int_0^t m(\tau)\,V_0(\tau)\,e^{\left[-\nu_i(t-\tau) - \int_\tau^t m(\eta)d\eta\right]}d\tau. \tag{77}$$

If, for example, the output signal corresponds to the local reaction of the system in point $x = 0$, the solution becomes:

$$\upsilon(0,t) = \frac{4}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i-1}\int_0^t m(\tau)V_0(\tau)\exp\left[-\nu_i(t-\tau) - \int_\tau^t m(\eta)d\eta\right]d\tau. \tag{78}$$

It is easy to observe that each term in (78) is similar to the expression for the solution (65) corresponding to the MS with the concentrated first-order parameters described by equation (64). To use the results obtained earlier, write the mathematical expectation expression for the solution (78) as:

$$\bar{\upsilon}(0,t)=\frac{4}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i-1}\int_0^t e^{-\nu_i(t-\tau)}M\left[m(\tau)V_0(\tau)\cdot e^{-\mu(\tau)}\right]d\tau, \tag{79}$$

where $\mu(\tau)=\int_\tau^t m(\eta)\,d\eta$.

Now, note that the expression for the mathematical expectation on the right side (79) matches in form with the expression for the mathematical expectation on the right side (66). Therefore, using the exact results obtained earlier, we can immediately write an expression for the mathematical expectation of the output signal. $\bar{\upsilon}(0,t)$.Based on the ratio (68), it has the form:

$$\bar{\upsilon}(0,t)=\frac{4}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i-1}\left\{\int_0^t\left[\bar{V}_0\bar{m}-\bar{V}_0\cdot\int_0^\tau k_m(\eta)d\eta+k_{m\theta}(0)+\right.\right.$$

$$\left.+\left(\int_0^\tau k_m(\eta)d\eta\right)\left(\int_0^\tau k_{\theta m}(\eta)d\eta\right)-\bar{m}\int_0^\tau k_{\theta m}(\eta)d\eta\right]\cdot e^{-(\nu_i+\bar{m})\tau+\tau\int_0^\tau k_m(\eta)d\eta-\int_0^\tau \eta k_m(\eta)d\eta}d\tau. \tag{80}$$

Consider the specific cases arising from the overall ratio (80).

Assume the measuring system be stationary, i.e. $m(t)\equiv\bar{m},\ k_m(\tau)=k_{m\theta}(\tau)\equiv 0.$ Then from (80) we get:

$$\bar{\upsilon}(0,t)=\frac{4}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i-1}\cdot\frac{\bar{m}\bar{V}_0}{\bar{m}+\nu_i}\cdot\left[1-e^{-(\bar{m}+\nu_i)t}\right]. \tag{81}$$

Since $\bar{\upsilon}(0,t)=\bar{u}(0,t)-u_0,\ \bar{V}_0(t)=\bar{\theta}(t)-u_0,$ it follows from (81) that, even in the absence of parametric effects, there is a systematic bias between the mathematical expectations of the MS $\bar{u}(0,t)$ readings and the measured signal $\bar{\theta}(t)$ in the steady state, due to the peculiarity of the physical model of this type of MS. The aforementioned feature of these MS's was also noted in the deterministic analysis in paragraph 1.4.

If there is no correlation between the parameter *m*(*t*) and the measured signal θ(*t*), from relation (80) it follows:

$$\bar{\upsilon}(0,t)=\frac{4}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i-1}\cdot\int_0^t\left[\bar{V}_0\bar{m}-\bar{V}_0\cdot\int_0^\tau k_m(\eta)d\eta\right]\times$$

$$\times\exp\left[-(\bar{m}+\nu_i)\tau+\tau\cdot\int_0^\tau k_m(\eta)d\eta-\int_0^\tau \eta k_m(\eta)d\eta\right]d\tau. \tag{82}$$

Based on the expressions (81), (82), it is possible to estimate the value of the parametric displacement between $\bar{u}(0,t)$ and $\bar{\theta}(t)$ the variable parameter *m*(*t*), and taking into account the total ratio (80), it is possible to estimate parametric displacement, due only to the correlation of the parameter *m*(*t*) and the measured signal θ(*t*).

Statistical analysis of dynamics of other types of MS with distributed parameters, including approximate methods, is given in [1].

5.4. PRINCIPLE OF INVARIANCE IN STATISTICAL DYNAMICS OF MEASURING SYSTEMS

The above fragments from the results of the statistical analysis of measuring systems, indicate that the presence of parametric effects in the dynamics of the MS significantly changes the idea of the processes taking place under real measurement conditions. and the quality of measurements.

In regard to this, as in the case of deterministic measurement processes, there is a problem of exclusion of the influence of parametric effects on the quality of measurements, in conditions where the measured signal and MS parameters are random processes. Since in statistical measurements the main task is usually to determine the mathematical expectation and variance of the measured signal, the problem that arises means the need to find such an algorithm of recovery of the mathematical expectation and dispersion of the measured signal, which would be invariant to the statistical properties of random parameters of the measuring system.

The use of the physically single-channel invariance principle to solve the indicated problem requires the construction of certain relations establishing the relationship between the main sought quantities, the statistical characteristics of the parameters of the measuring system, the influence of which should be excluded, and the statistical characteristics of the MS readings.

Thus, from the above it follows that the use of the single-channel principle of invariance necessitates the transition from the original mathematical model of MS, given usually in the form of a differential or integral equation establishing the relationship between MS $Y(t)$ and the measured signal $X(t)$, to some statistical model that establishes the relationship between the statistical characteristics of the signals $Y(t)$, $X(t)$ and the MS's parameters. As can be seen from the contents of the previous paragraphs of this chapter, there are various exact and approximate methods of this transition.

Any method of transition to statistical models can also be accompanied by the use of existing decomposition methods for random functions, correlation and mutual correlation functions, including, of course, the widely known and deeply developed canonical decomposition method. It is important, however, in all cases to bear in mind that the resulting statistical models should be linear in relation to the main sort, or linearized by the introduction of auxiliary desired values.

It follows from the previous presentation that, from the point of view of the invariance principle, the most effective method for this transition to the statistical MS model is the on the theory of Markov processes. It is connected both with the commonality and accuracy of the method, and with the form in which the statistical model of IP is obtained: the model is linear or easily linearized relative to the desired values. Therefore, the following results relate to the construction of a statistical MS model using Markov process theory methods.

Consider the first MS model with concentrated parameters, assuming $n = 1$. In this case, the reference MS model is as follows:

$$\frac{dY(t)}{dt} + a(t)Y(t) = X(t), \quad Y(0) = 0. \tag{83}$$

Consider two of the possible measurement situations:

1 - input signal $X(t)$ and system parameter $a(t)$ are normally distributed stationary uncorrelated white noise;

2 - The input signal $X(t)$ and the system parameter $a(t)$ are normally distributed stationary and stationary correlated white noise.

1. Input signal *X*(*t*) and system parameter *a*(*t*) – normally distributed stationary uncorrelated white noise.

As can be seen from (20) and (22), in this case the statistical model of the MS is the equations:

$$\dot{m} + (C - \frac{1}{2}G^V)m = m_x, \quad m(0) = 0 \tag{84}$$

$$\dot{D} + 2(C - G^V)D = G^N, \quad D(0) = 0 \tag{85}$$

The first of these equations is the equation for the mathematical expectation of MS readings, and the second for the variance of MS readings. All symbols in equations (84), (85) have the same meaning as in p. 5.1.

So, in this case, in accordance with the invariance principle, it is necessary to build such an algorithm for restoring the statistical characteristics of the measured signal - the mathematical expectation m_x and G^N intensity, which would be invariant to the parametric effect, i.e. invariant to the values of unknown values: C - the mathematical expectation of the parameter $a(t)$ and G^V - the intensity of this parameter.

By analogy with the case of deterministic signals and parameters, the application of the invariance principle for randomly measured signals and MS parameters is based on the transition to an extended measurement problem, but now to an extended statistical measurement problem.

Assume $\tilde{m}_x, \tilde{G}^N, \tilde{C}, \tilde{G}^V$ - estimates of m_x, G^N, C, G^V values, respectively, which are expected to be found in the process of restoring m_x, G^N values.

According to (84), (85), we have two residuals in the differential form $I_{m\,dif}(t)$, $I_{D\,dif}(t)$:

$$I_{m\,dif}(t) = \dot{m} + (\tilde{C} - \frac{1}{2}\tilde{G}^V)m - \tilde{m}_x,$$

$$I_{D\,dif}(t) = \dot{D} + 2(\tilde{C} - \tilde{G}^V)D - \tilde{G}^N, \tag{86}$$

the first of which characterizes the residual by mathematical expectation, the second - by variance.

Since, according to the method of constructing the invariance algorithm, along with the basic unknown values of $\tilde{m}_x, \tilde{G}^N$, the characteristic of the parameter $a(t)$, i.e $\tilde{C}, \tilde{G}^V$, should also be included in the number of unknown, the total number of unknown turns out to be equal to 4.

Let us introduce a single designation of unknown:

$$X_1 = \tilde{C} - \frac{1}{2}\tilde{G}^V, \quad X_2 = \tilde{m}_x, \quad X_3 = 2(\tilde{C} - \tilde{G}^V), \quad X_4 = \tilde{G}^N.$$

Then the residuals will take the form:

$$I_{m\,dif}(t) = \dot{m} + m(t)\cdot X_1 - X_2, \tag{87}$$

$$I_{D\,dif}(t) = \dot{D} + D(t)\cdot X_3 - X_4. \tag{88}$$

Since there are now two residuals in the statistical dynamics of the MS, each one needs to build its own basic SLAE. Assume, considering the mathematical expectation and variance of the MS readings known and using any of the methods of constructing the basic SLAE considered in chapter 3, for each of the residuals $I_{m\,dif}(t)$, $I_{D\,dif}(t)$ built and solved its SLAE. Having thus found the values of X_1, …, X_4, we return to the former unknown. Obviously, the main unknown $\tilde{m}_x$, $\tilde{G}^N$ are defined by the expressions $\tilde{m}_x = X_2, \tilde{G}^N = X_4$. Additional unknown $\tilde{C}, \tilde{G}^V$ are from the system:

$$\tilde{C} - \frac{1}{2}\tilde{G}^V = X_1,$$

$$2(\tilde{C} - \tilde{G}^V) = X_3,$$

from where we have

$$\tilde{C} = \frac{4X_1 - X_3}{2}, \quad \tilde{G}^V = 2X_1 - X_3.$$

Thus, the algorithm of recovery of statistical characteristics m_x, G^N of the measured signal is constructed, and this algorithm is invariant to the values of C, G^V, characterizing the statistical properties parameter $a(t)$ of the measuring system.

2. Input signal $X(t)$ and system parameter $a(t)$ – normally distributed stationary and stationary correlated white noises.

Consider the more complex situation of measurements: assume the correlation condition of the measured signal $X(t)$ and parameter $a(t)$ are added to the conditions considered in the previous case. Then, when moving from the original model (83) to the statistical model for the mathematical expectation of the output signal $m(t)$, according to (20), we have:

$$\dot{m} + \left(C - \frac{1}{2}G^V\right)m = m_x - \frac{1}{2}G^{VN}, \quad m(0) = 0. \tag{89}$$

In the above mentioned work [6] the following expression of the variance of the output signal is given for the system under consideration.

$$D(t) = \frac{G^N}{2(C - G^V)}\left[1 - e^{-2(C-G^V)t}\right] + 2G^{VN}\,\frac{m_x - 0{,}5G^{VN}}{C - 0{,}5G^V}\times$$

$$\times\left\{\frac{1}{2(C - G^V)}\left[1 - e^{-2(C-G^V)t}\right] - \frac{1}{\tilde{N} - \frac{3}{2}G^V}\left[e^{-(C-0{,}5G^V)t} - e^{-2(C-G^V)t}\right]\right\}. \tag{89a}$$

It is easy to see by direct substitution that this expression is the solution of the differential equation:

$$\dot{D} + 2(C - G^V)D = G^N + 2G^{VN}\cdot m(t). \tag{90}$$

Thus, the statistical model for this measurement situation is a set of equations (89) - for the mathematical expectation of the MS readings and (90) - for the variance of the MS readings in which the processes $m(t)$, $D(t)$, relating to the MS readings are considered, naturally, known.

Now, in accordance with the principle of invariance, it is necessary to build such an algorithm of recovery of statistical characteristics of the measured signal - mathematical expectation m_x and intensity G^N, which would be invariant to the values of unknown values: C - the mathematical expectation of the parameter $a(t)$, G^V – the intensity of this parameter, and G^{VN} – the mutual intensity of the parameter $a(t)$ and the measured signal $X(t)$.

If the estimate of the G^{VN} value is defined as $\tilde{G}^{VN}$, similar to (86) we now have the following expression of residuals in differential forms for the mathematical expectation and variance of MS readings:

$$I_{m\,dif}(t) = \dot{m} + \left(\tilde{C} - \frac{1}{2}\tilde{G}^V\right)m - \tilde{m}_x + \frac{1}{2}\tilde{G}^{VN} \tag{91}$$

$$I_{D\,dif}(t) = \dot{D} + 2(\tilde{C} - \tilde{G}^V)D - \tilde{G}^N - 2\tilde{G}^{VN}m(t) \tag{92}$$

According to the method of constructing invariance algorithms, in the number of unknown, along with the basic values, $\tilde{m}_x, \tilde{G}^N$, we will also include the values $\tilde{C}, \tilde{G}^V, \tilde{G}^{VN}$. Therefore, the total number of unknown is now equal to 5. Let us introduce a single designation of unknown:

$$X_1 = \tilde{C} - \frac{1}{2}\tilde{G}^V;\ X_2 = \tilde{m}_x - \frac{1}{2}\tilde{G}^{VN};\ X_3 = 2(\tilde{C} - \tilde{G}^V);\ X_4 = \tilde{G}^N;\ X_5 = 2\tilde{G}^{VN},$$

then residuals expressions will take the form:

$$I_{m\,dif}(t) = \dot{m} + m(t) \cdot X_1 - X_2, \tag{93}$$

$$I_{D\,dif}(t) = \dot{D} + D(t)X_3 - X_4 - m(t)X_5. \tag{94}$$

Having constructed by any method the main SLAE for each of these residuals and solving these independent SLAEs, we obtain the values of unknown quantities $X_1, \ldots, X_5$.

The solution of the basic SLAE for the desired X_3, X_4, X_5 gives:

$$\tilde{G}^N = X_4,\ \tilde{G}^{VN} = \frac{1}{2}X_5,\ 2(\tilde{C} - \tilde{G}^V) = X_3.$$

The solution of the basic SLAE for the desired X_1, X_2 gives:

$$\tilde{C} - \frac{1}{2}\tilde{G}^V = X_1,\quad \tilde{m}_x - \frac{1}{2}\tilde{G}^{VN} = X_2 \ \Rightarrow\ \tilde{m}_x = X_2 + \frac{1}{4}X_5.$$

Finally, considering the expressions for X_1 and X_3, together, we find:

$$\tilde{C} = \frac{4X_1 - X_3}{2},\quad \tilde{G}^V = 2X_1 - X_3.$$

Thus, estimates of both the main sought m_x, G^N, characterizing the statistical properties of the measured signal, and estimates of additional searched C, G^V, G^{VN}, characterizing the statistical properties of the parameter of the measuring system, as well as the degree of correlation of this parameter with the measured signal. This is the construction of an algorithm for restoring the statistical characteristics of the measured signal $X(t)$, which is invariant to parametric effects caused by the statistical nature of the change in the parameter $a(t)$ of the measuring system.

Model implementation of the algorithm of recovery of statistical characteristics of the measured signal

The following are the results of the simulation of the recovery of the statistical characteristics of the measured signal related to the second, more general, measurement situation, in which the measured signal $X(t)$ and parameter $a(t)$ are correlated. For modeling, it is necessary to choose the method of compiling the main SLAE, as well as direct and indirect criteria for assessing the accuracy of the obtained results.

As with the method of compiling the basic SLAE, select the second of the methods described in chapter 3 (analogous to the method of collocation). According to this method, when compiling the basic SLAE to determine the unknown X_3, X_4, X_5, it is necessary to require equal zero residual $I_{D\,dif}(t)$ at three points. If, for example, the recovery of the desired values starts at the point $t = t_1$ and uses some p-th recovery interval having the duration $T0P$, these points will be $t_k = t_1 + TP(k-1)$, $k = 1, 2, 3$, $TP = T0P/2$, and the basic SLAE for the definition of X_3, X_4, X_5 is as follows:

$$\sum_{i=1}^{3} A_{ki} X_{i+2} = B_k,\quad k = 1, 2, 3$$

$$A_{k1} = D(t_k),\ A_{k2} = -1,\quad A_{k3} = -m(t_k),\quad B_k = -\left(\frac{d}{dt}D(t)\right)\bigg|_{t=t_k}$$

Similarly, the basic SLAE for the determination of the values of X_1, X_2, is constructed, using the residual $I_{m\,dif}(t)$, and the recovery interval $T0P$ remains the same as in the previous case, but $TP = T0P/1 = T0P$.

In solving the basic SLAE as the mathematical expectation $m(t)$ and the variance $D(t)$ of output signals the previously mentioned solutions of direct problems (89), (90) respectively are used.

The following simple criteria are used as direct criteria for assessing the accuracy of the obtained results:

$$\delta_1 = \frac{|m_x - \tilde{m}_x|}{|m_x|},\ \delta_2 = \frac{|G^N - \tilde{G}^N|}{G^N},\ \delta_3 = \frac{|\tilde{N} - \tilde{\tilde{N}}|}{\tilde{N}},$$

$$\delta_4 = \frac{\left|G^V - \tilde{G}^V\right|}{G^V}, \ \ \delta_5 = \frac{\left|G^{VN} - \tilde{G}^{VN}\right|}{\left|G^{VN}\right|}.$$

As an indirect criterion for the proximity of the obtained estimates to estimated values, we can use both criteria based on the mathematical expectation of the MS readings, and criteria based on the variance of the MS readings. The first of them is used below, and as a specific type of indirect criterion we choose the following:

$$\rho I_m = \frac{1}{T0P} \int_{t_1}^{t_1+T0P} \left|m(t) - M(t)\right| dt, \quad \rho I_{m\,\text{от}} = \frac{\rho I_m}{\frac{1}{T0P} \int_{t_1}^{t_1+T0P} \left|m(t)\right| dt}, \tag{95}$$

where $T0P$ – pth interval of recovery of the desired values; $m(t)$ – the actual value of the mathematical expectation of the MS readings, i.e. the value represented by the expression (21); $M(t)$ – the estimation of the mathematical output expectations (the estimated value of the output mathematical expectation), that is, it is the mathematical expectation of the output signal that would have occurred if in the expression (21) instead of true values of m_x, C, G^V, G^{VN} were found estimates $\tilde{m}_x$, $\tilde{C}$, $\tilde{G}^V$, $\tilde{G}^{VN}$.

Since the evaluation starts at the point $t = t_1$, the intended solution is taken with the initial condition at $t = t_1$, and the initial value $M(t_1)$ is taken equal to the value $m(t)$ at the point $t = t_1$ (the same as the recovery of deterministic signals in Chapter 3). Thus, the expected value of the mathematical expectation of the output signal is determined by the expression:

$$M(t) = m(t_1) \cdot e^{-(\tilde{C} - \frac{1}{2}\tilde{G}^V)(t - t_1)} + \frac{\tilde{m}_x - \frac{1}{2}\tilde{G}^{VN}}{\tilde{C} - \frac{1}{2}\tilde{G}^V} \cdot \left[1 - e^{-(\tilde{C} - \frac{1}{2}\tilde{G}^V)(t - t_1)}\right]. \tag{96}$$

In the third chapter, when recovering time-varying deterministic signals, we used the $\rho I_{u\,\text{от}}$, criterion, which differs from the criterion given above $\rho I_{m\,\text{от}}$ only by the fact that in it, instead of mathematical expectations, $m(t)$, $M(t)$ of the output signals, the output signals $u(t)$, $U(t)$ were used. It was noted that the recovery interval $[t_1, t_1 + T0P]$ in the area of the established stage of measurement was accompanied by a sharp increase in the values of this criterion, and a corresponding increase in the errors in recovering the measured signals. The peculiarity of the previously considered deterministic signals (except for the first typical signal) was that in the established stage of measurement by the influence of initial conditions on values $u(t)$, $U(t)$ can be neglected, but the values of the actual $u(t)$ and the assumed $U(t)$ of the output signals continued to be different and time-dependent functions. Due to this, the dependence between the values of the $\rho I_{u\,\text{от}}$ criterion, and the accuracy of recovery of the measured signal, continued.

As follows from the expressions (21), (96), for a stable measuring system, the mathematical expectations $m(t)$, $M(t)$ in the steady stage of measurement are constant values. Therefore, at this stage of the process of restoring the statistical characteristics of the measured signal, the value of the $\rho I_{m\,\text{от}}$ criterion is constant, close to zero, while error recovery is growing. This means that at the specified stage of measurement, the correlation of this criterion with the error of the obtained estimates is lost. Thus, the criterion $\rho I_{m\,\text{от}}$ is important only for the transition stage, where only its values indirectly characterize the proximity of the found estimates to the evaluated parameters.

Since it is a question of restoring the statistical characteristics of stationary random signals, that is, characteristics that are constant values, it is sufficient to evaluate these characteristics at the first interval t_1, $t_1 + T01]$, and their estimates at subsequent intervals $[t_1, t_1 + + T0P]$, $p = 2, 3, \ldots$ can only confirm the correctness of the information obtained at the first interval. Once the estimates at the pth interval are very different from their values at previous intervals, this would indicate that a significant part of the pth interval is already the established stage of measurement. Therefore, the process of restoring the desired statistical characteristics can be considered completed at (p - 1)-th interval.

Generally speaking, it is possible to introduce a simple criterion $\Delta_{\text{от}}$ which, at each interval of receiving estimates, characterizes the degree of achievement of the established stage of measurement:

$$\Delta_{\text{от}} = \frac{|m\,[t_1 + \text{TP}(k-1)] - m\,[t_1 + \text{TP}(k-2)]|}{|m\,[t_1 + \text{TP}(k-1)]|}, \text{TP} = \frac{T0P}{k-1},$$

where k is the number of unknown in the main SLAE.

This expression represents the relative difference between the values of the mathematical expectation of the MS at the last and penultimate points of collocation of the pth recovery interval. In this case, for example, for the main SLAE containing unknown X_3, X_4, X_5, values $k = 3$, so

$$\Delta_{\text{от}} = \frac{|m(t_1 + 2\text{TP}) - m(t_1 + \text{TP})|}{|m(t_1 + 2\text{TP})|}.$$

Obviously, the lower the value of the criterion $\Delta_{\text{от}}$, the higher the degree of achievement of the established stage of measurement.

Tables 1 and 2 show the results of restoring the statistical characteristics of the measured signals under conditions where the measured signal $X(t)$ and parameter $a(t)$ are normally distributed stationary and stationary correlated white noises.

Values from $\Delta_{\text{от}}$, $\rho I_{m\ \text{от}}$, δ_1, …, δ_5 given in the tables are given in percentages. The specified numbers of conditionalities belong to the main SLAE containing the desired X_3, X_4, X_5. Conditionality numbers related to the main SLAE containing the sought X_1, X_2, are not given, as they are significantly smaller than for the previous basic SLAE.

The results in Table 1 correspond to the process described above to the statistical recovery of the measured signal. Namely up to the sixth interval of recovery of the desired characteristics, the values of the received estimates almost coincide with the values of the estimated characteristics.

Errors appear on the seventh recovery interval, but the maximum error (δ_4), related to the G^V, parameter is only 0.014%, which can be considered a negligible value for statistical studies. But already on the next (eighth), recovery interval values differ sharply from the values of these estimates at the previous intervals: these differences range from 7.7% for m_x parameter to 393% for G^V parameter. Therefore, the process of recovering the desired characteristics using the invariance algorithm should be completed at the seventh interval.

Note that the value of the criterion $\Delta_{\text{от}}$ the seventh interval is only 0.003%, and this also indicates that the seventh recovery interval, which ends using the invariance, partially covers the established stage of measurement. At the eighth interval, the value of this criterion is almost zero, that is, the eighth recovery interval largely covers the established stage of measurement.

Finally, the results in Table 1 related to the ninth recovery interval show that this interval consists mainly of an established measurement stage, so the value of the criterion $\Delta_{\text{от}}$ is zero, and the main SLAE becomes singular.

Let us refer to the results of the model implementation of the invariance algorithm presented in Table 2.

Table 1. **The results of recovery of statistical characteristics of the measured signal and the parameters of the MS.**

Initial data $m_x = 100$, $G^N = 40$, C = 10, $G^V = 4$, $G^{VN} = 8$

P	$\Delta_{от}$	$t_1 \le t \le t_1 + T0P$	TP	$T0P$	Indirect criterion $\rho I_{m\ от}$	Direct criteria					cond 1 (A) cond 2 (A)
						$\delta_1 \cdot 10^6$	$\delta_2 \cdot 10^6$	$\delta_3 \cdot 10^6$	$\delta_4 \cdot 10^6$	$\delta_5 \cdot 10^6$	
1	19,36	[0,04; 0,08]	0,02	0,04	0,002	0	0	0	0	0	382 217
2	28,4	[0,04; 0,12]	0,04	0,08	0,002	0	0	0	0	0	177 116
3	22,68	[0,04; 0,2]	0,08	0,16	0,001	0	0	0	0	0	120 89
4	15,44	[0,04; 0,36]	0,16	0,32	0	0	0	0	0	0	135 100
5	5,2	[0,04; 0,68]	0,32	0,64	0	0	0	0	0	0	305 219
6	0,43	[0,04; 1,32]	0,64	1,28	0	0	0	0	0	0	3015 2229
7	0,003	[0,04; 2,6]	1,28	2,56	0	277	5984	2823	14118	6933	467645 347818
8	$9{,}3 \cdot 10^{-8}$	[0,04; 5,16]	2,56	5,12	0	$7{,}7 \cdot 10^6$	$167 \cdot 10^6$	$78{,}6 \cdot 10^6$	$393 \cdot 10^6$	$193 \cdot 10^6$	$13 \cdot 10^9$ $9{,}7 \cdot 10^9$
9	0	[0,04; 10,28]	5,12	10,24	–	–	–	–	–	–	singularity

Table 2. **The results of recovery of statistical characteristics of the measured signal and the parameters of the MS.**

Initial data $m_x = 100$, $G^N = 40$, C = 10, $G^V = 0.4$, $G^{VN} = 3$

P	$\Delta_{от}$	$t_1 \le t \le t_1 + T0P$	TP	$T0P$	Indirect criterion $\rho I_{m\ от}$	Direct criteria					cond 1 (A) cond 2 (A)
						$\delta_1 \cdot 10^6$	$\delta_2 \cdot 10^6$	$\delta_3 \cdot 10^6$	$\delta_4 \cdot 10^6$	$\delta_5 \cdot 10^6$	
1	24,4	[0,04; 0,08]	0,02	0,04	0,009	0	0,001	0,002	0,102	0,005	427 217
2	32,3	[0,04; 0,12]	0,04	0,08	0,006	0	0	0,003	0,016	0,001	174 102
3	38,1	[0,04; 0,2]	0,08	0,16	0,004	0	0	0,001	0,004	0	89 76
4	40,9	[0,04; 0,36]	0,16	0,32	0,002	0	0	0	0	0	103 94
5	40,36	[0,04; 0,68]	0,32	0,64	0,001	0	0	0	0	0	199 166
6	35,66	[0,04; 1,32]	0,64	1,28	0,001	0	0	0	0	0	423 333
7	25,47	[0,04; 2,6]	1,28	2,56	0	0	0	0	0	0	802 604
8	11,1	[0,04; 5,16]	2,56	5,12	0	0	0	0	0	0	1179 875
9	1,59	[0,04; 10,28]	5,12	10,24	0	0	0	0	0	0	1901 1804
10	0,027	[0,04; 20,52]	10,24	20,48	0	0	0	0	0,001	0	106374 84104
11	$7 \cdot 10^{-6}$	[0,04; 41]	20,48	40,96	0	0,126	4,418	4,647	23,236	8,404	$3{,}8 \cdot 10^8$ $3 \cdot 10^8$
12	$6 \cdot 10^{-13}$	[0,04; 81,96]	40,96	81,92	0	$1{,}7 \cdot 10^6$	$59{,}5 \cdot 10^6$	$62{,}6 \cdot 10^6$	$313 \cdot 10^6$	$113 \cdot 10^6$	$4{,}9 \cdot 10^{15}$ $3{,}9 \cdot 10^{15}$
13	0	[0,04; 163,88]	81,92	163,84	–	–	–	–	–	–	singularity

Table 2 shows the results of the recovery of the statistical characteristics of the measured signal for the measuring systems, which are approximately an order of magnitude more inertial than in the previous case. We will not give a qualitative analysis of the process of restoring the desired characteristics for this case, as this analysis is quite similar to the analysis above . Note only that, according to the results given in Table 2, now the last interval at which the recovery of the desired characteristics using the invariance algorithm is appropriate, is the eleventh interval (p = 11). As we can see now, due to the greater inertia of the measuring system, the length of the interval at which the effective use of the invariance algorithm is possible has increased ~16 times.

The general conclusion arising from the results of the model implementation of the invariance algorithm is that the use of the single-channel principle of invariance allowed us, with a very high accuracy and invariant to parametric effects, to restore the statistical characteristics of the measured signal.

In conclusion, we will address the question of the preliminary stage of implementation of the principle of invariance. When implementing the invariance algorithm for deterministic measuring systems and measured signals, the preliminary stage consisted of choosing mathematical models of estimates $\tilde{a}(t)$, $\tilde{X}(t)$. In the simplest case, it is a choice for estimating $\tilde{a}(t)$, $\tilde{X}(t)$ degrees of approximating algebraic polynomials and the specified choice is made using an indirect criterion.

A similar stage takes place in the case of implementation of the principle of invariance in the restoration of statistical characteristics of the measured signal. In the parsed simple examples, it was not necessary to use approximations to estimate the desired characteristics of random processes. Therefore, for the studied models of measuring systems, the content of the preliminary stage consists of the selection of the specific, most adequate statistical model of the measuring system. Since it is not known apriori whether the statistical models (84), (85) or (89), (90) most adequately reflect the measurement situation that actually occurs, it is necessary to test both of these models.

There may be a situation in which none of the cases considered is satisfactory. This situation means that it is necessary to refer to the specification of either the original MS model on which the statistical models are built, or the method of transition from the source model to the statistical MS model. It is clear that all the above steps, including model refinements, can be performed using a single program.

On the evaluation of the error of recovery of the measured signal in real conditions of measurement

When using the single-channel principle of invariance in real conditions of measurement, an important question arises about estimation of error of recovery of the measured signal using direct criteria. Direct criteria, in one form or another, use the concept of deviation $|\tilde{X}(t)–X(t)|$ of the measured signal $X(t)$ from its estimate $\tilde{X}(t)$. In modeling the signal $X(t)$ is known a priori, and its estimate $\tilde{X}(t)$ is obtained in the process of recovering the measured signal using the invariance algorithm. Therefore, when calculating the specified deviation in the simulation process, there are no questions.

When using the single-channel principle of invariance in real conditions of measurement the $X(t)$ signal is unknown and therefore the problem of estimating this deviation arises.

One possible way of doing this is set out below.

Assume the first order MS described by equation (3.2) be considered, and the recoverable processes $\alpha(t)$, $X(t)$ are represented by algebraic polynomials

$$\alpha(t) = \sum_{i=1}^{N1} \alpha_i t^{i-1}, \qquad X(t) = \sum_{j=1}^{N2} \beta_j t^{j-1}$$

The equation (3.2) is written as

$$0 = \frac{du(t)}{dt} \sum_{i=1}^{N1} \alpha_i t^{i-1} + u(t) - \sum_{j=1}^{N2} \beta_j t^{j-1}, \tag{a}$$

where the values of $N1$, $N2$, α_i, β_j are estimated.

The residual in the differential form $I_{dif}(t)$ is described by the expression

$$I_{dif}(t) = \frac{du(t)}{dt} \sum_{i=1}^{n1} \tilde{\alpha}_i t^{i-1} + u(t) - \sum_{j=1}^{n2} \tilde{\beta}_j t^{j-1}, \tag{b}$$

where $\tilde{\alpha}_i$, $\tilde{\beta}_j$ – estimates of α_i, β_j values respectively.

Assume that process recovery task $\alpha(t)$, $X(t)$ has been solved. This means that the values of $n1$, $n2$, $\tilde{\alpha}_i$, $\tilde{\beta}_j$, and therefore estimates $\tilde{\alpha}(t)$, $\tilde{X}(t)$,, are found, and, by the method of selecting the only one of the many possible solutions of the inverse problem, it is established: $n1 = N1$, $n2 = N2$. It remains to find deviations $|\tilde{\alpha}(t) - \alpha(t)|$, $|\tilde{X}(t) - X(t)|$ and corresponding direct estimates of errors in recovery processes $\alpha(t)$ and $X(t)$.

Since estimates $\tilde{\alpha}(t)$, $\tilde{X}(t)$ are already known, the expression (b) now allows us to calculate $I_{dif}(t)$ at an arbitrary point in the recovery interval.

Subtracting expression (a) from expression (b) and considering that $n1 = N1$, $n2 = N2$, we have

$$\frac{du(t)}{dt}\sum_{i=1}^{N1}(\Delta\alpha)_i t^{i-1} - \sum_{j=1}^{N2}(\Delta\beta)_j t^{j-1} = I_{dif}(t), \tag{c}$$

where: $(\Delta\alpha)_i = (\tilde{\alpha}_i - \alpha_i)$, $(\Delta\beta)_j = (\tilde{\beta}_j - \beta_j)$.

At the same time, the deviations themselves have the form of

$$\Delta\alpha(t) = [\tilde{\alpha} - \alpha(t)] = \sum_{i=1}^{N1}(\Delta\alpha)_i t^{i-1}, \quad \Delta X(t) = [\tilde{X}(t) - X(t)] = \sum_{j=1}^{N2}(\Delta\beta)_j t^{j-1}$$

Thus, finding the values $(\Delta\alpha)_i$, $(\Delta\beta)_j$, it is possible to find the desired deviations.

The ratio (c) can be considered as an equation establishing a relationship between the known functions $u(t)$, $I_{dif}(t)$ and unknown constants $(\Delta\alpha)_i$, $(\Delta\beta)_j$. That is, the situation here is quite similar to the one that occurred earlier in the construction of the invariance algorithm. Therefore, to determine the values $(\Delta\alpha)_i$, $(\Delta\beta)_j$, by analogy with the construction of the basic SLAE, let us construct SLAE for errors.

Let us introduce a single designation of unknown. Assume, that

$$(\Delta\alpha)_i = \tilde{X}_i,\ i = 1, 2, \ldots, N1, \quad (\Delta\beta)_j = \tilde{X}_{N1+j},\ j = 1, 2, \ldots, N2$$

Then the ratio (c) is written as

$$\frac{du(t)}{dt}\sum_{i=1}^{N1} t^{i-1}\cdot\tilde{X}_i - \sum_{i=N1+1}^{N1+N2} t^{i-N1-1}\cdot\tilde{X}_i = I_{dif}(t)$$

When building SLAE for errors, you can use the same methods as for building basic SLAE. Using, for example, the second method requires satisfying the ratio (c) at the points $t'_k = t'_1 + T'(k-1)$, $k = 1, 2, \ldots, (N1 + N2)$, где $T' = T'0/(N1 + N2 - 1)$, $T'0$ – the interval of error estimation, and T' - partial interval.

Satisfaction of these conditions gives SLAE for errors, which in standard form has the form

$$\sum_{i=1}^{N1+N2}\tilde{A}_{ki}\tilde{X}_i = \tilde{B}_k, \quad \tilde{B}_k = I_{dif}[t'_1 + T'(k-1)], \quad k = 1, 2, \ldots, (N1+N2) \tag{d}$$

$$\tilde{A}_{ki} = \begin{cases} \left.\dfrac{du(t)}{dt}\right|_{t=t'_1+T'(k-1)}\cdot[t'_1 + T'(k-1)]^{i-1}, & i \le N1 \\ -[t'_1 + T'(k-1)]^{i-N1-1}, & i > N1 \end{cases}$$

As we can see, the SLAE matrix for errors (d) at $N1 = n1$, $N2 = n2$ coincides in structure with the matrix of the main SLAE (3.8) to determine estimates $\tilde{\alpha}(t)$, $\tilde{X}(t)$. Note that SLAE (3.8) is built using points $t_k = t_1 + T(k-1)$, $k = 1, 2, \ldots, (N1 + N2)$, where $T = T0/(N1 + N2 - 1)$, $T0$– recovery interval $\alpha(t)$, $X(t)$, a T – partial interval. That is, the t'_k, points used in the preparation of SLAE for errors (d) are different from the t_k, points used in the compilation of the main SLAE. This was done in order to avoid uniformity of the SLAE (d), since, according to the meaning of constructing the main SLAE (3.8) at the points t_k the conditions $I_{dif}(t_k) = 0$ must be satisfied. It is clear that the selection of t'_k points is subject to a restriction: interval $[t'_1; t'_1 + T'0]$ should be included in interval $[t_1; t_1 + T0]$.

Thus, to estimate the errors of recovery processes $\alpha(t)$, $X(t)$ in real conditions measurement is enough, deciding the system (d), to find the values $(\Delta\alpha)_i$, $(\Delta\beta)_j$, and then deviations $\Delta\alpha(t)$, $\Delta X(t)$. Restoring errors $\alpha(t)$, $X(t)$ are calculated according to, for example, direct criteria

$$(\rho\alpha_{\text{от}})_{\text{в}} = \frac{\int_{t'_1}^{t'_1+T'0}\Delta\alpha(t)\,dt}{\int_{t'_1}^{t'_1+T'0}|\tilde{\alpha}(t)|\,dt}, \qquad (\rho X_{\text{от}})_{\text{в}} = \frac{\int_{t'_1}^{t'_1+T'0}\Delta X(t)\,dt}{\int_{t'_1}^{t'_1+T'0}|\tilde{X}|\,dt}$$

Here in denominators estimates are used, because in real conditions of measurement it is possible to estimate the desired values.

The following contains the methodology for estimating the errors of recovery of the processes $\alpha(t)$, $X(t)$ using the single-channel invariance principle in real measurement conditions. This technique can also be used for consideration of other types of MS, as well as for the random nature of processes $\alpha(t)$, $X(t)$, and regardless of the methods of constructing the main SLAEs.

Turn to numerical estimates, for which we will refer to the specific conditions of measurement. Assume the reference conditions for the MS are

$N1$	$N2$	α_1	α_2	α_3	α_4	β_1	β_2	β_3	t_1	$T0$
4	3	1	10	100	1,000	2	20	200	0.04	0.1

and let, as a result of the recovery of processes $\alpha(t)$, $X(t)$ by the single channel principle of invariance is established (see Table 3.4):

$$n1 = N1 = 4,\ n2 = N2 = 3,\ \rho\alpha_{\text{от}} = 1.307 \cdot 10^{-5}\,\%,\ \rho X_{\text{от}} = 1.269 \cdot 10^{-5}\,\%$$

These errors are defined in the assumption that processes $\alpha(t)$, $X(t)$, are known, as was the case in modeling. The values of these errors can be considered as "accurate" and used to compare them with estimates $(\rho\alpha_{\text{от}})_{\text{в}}$, $(\rho X_{\text{от}})_{\text{в}}$, obtained according to the above methodology.

If you use points as t'_k points: $t'_1 = 0.0405$, $T'0 = 0.0995$, $t'_k = t'_1 + T'(k-1)$, $T' = T'0/(N1 + N2 - 1)$, $k = 1, 2, \ldots, (N1 + N2)$, that is, by shifting the first point of the initial interval $t \in [0.04; 0.14]$ on 0.0005, take the interval $t' \in [\ 0.0405; 0.14]$, then using the found estimates $\tilde{\alpha}(t)$, $\tilde{X}(t)$, in accordance with the described methodology, we get

$$(\rho\alpha_{\text{от}})_{\text{в}} = 4.363 \cdot 10^{-3}\,\%, \quad (\rho X_{\text{от}})_{\text{в}} = 4.236 \cdot 10^{-3}\,\%.$$

Practically the same result is obtained if you move the last point of the initial interval to the left by 0.0005 or perform both of these shifts.

Two conclusions are drawn from the above. First, the solution of the given equation for the error of estimates (c) allows us to distinguish from many possibilities the only solution of the inverse problem, both in the conditions of real measurements and in the conditions of modeling. Second, even according to the overestimated errors $(\rho\alpha_{\text{от}})_{\text{в}}$, $(\rho X_{\text{от}})_{\text{в}}$, calculated for real measurement conditions, the recovery of the sought $\alpha(t)$, $X(t)$ using the single-channel principle invariance is indeed carried out with negligible error for dynamic measurements. Note that significantly more accurate estimates of errors $(\rho\alpha_{\text{от}})_{\text{в}}$, $(\rho X_{\text{от}})_{\text{в}}$ are obtained using the first method of constructing SLAE.

In conclusion, let's consider a more complex case. Assume the MS be described by equation (3.1)

$$\frac{du(t)}{dt} + a(t)u(t) = a(t)X(t), \qquad \text{(a')}$$

that is, now we are dealing with the second scheme of implementing the invariance algorithm.

By integrating (a′), we get the original model in an integral form

$$u(t) - u(t_1) + \int_{t_1}^{t} a(\tau)u(\tau)d\tau - \int_{t_1}^{t} a(\tau)X(\tau)d\tau = 0 \qquad \text{(b')}$$

The residual in the integral form is described by the expression

$$I_{\text{int}}(t) = u(t) - u(t_1) + \int_{t_1}^{t} \tilde{a}(\tau)u(\tau)\,d\tau - \int_{t_1}^{t} \tilde{a}(\tau)\tilde{X}(\tau)\,d\tau, \qquad \text{(c')}$$

where $\tilde{a}(t)$, $\tilde{X}(\tau)$ – known estimates of unknown $a(t)$, $X(t)$, found in accordance with the second scheme of implementation of the invariance algorithm described in section 3.3.2.

Residual $I_{\text{int}}(t)$ is now also known, since estimates $\tilde{a}(t)$, $\tilde{X}(t)$ are known.

Subtract the expression (c′) from the expression (b′), we get

$$I_{\text{int}}(t) = u(t) - u(t_1) + \int_{t_1}^{t} \tilde{a}(\tau)u(\tau)\,d\tau - \int_{t_1}^{t} \tilde{a}(\tau)\tilde{X}(\tau)\,d\tau,$$

By denoting $\Delta a(t) = \tilde{a}(t) - a(t)$, $\Delta X(t) = \tilde{X}(t) - X(t)$ and using the presentation

$$\tilde{a}(t)\tilde{X}(t) - a(t)X(t) = \Delta a(t)\cdot\tilde{X}(t) + \tilde{a}(t)\Delta X(t) - \Delta a(t)\cdot\Delta X(t),$$

get the desired equation for estimates errors in the form of

$$\int_{t_1}^{t}[u(\tau)-\widetilde{X}(\tau)]\cdot\Delta a(\tau)d\tau-\int_{t_1}^{t}\widetilde{a}(\tau)\Delta X(\tau)d\tau+\int_{t_1}^{t}\Delta a(\tau)\Delta X(\tau)d\tau=I_{\text{int}}(t) \qquad (d')$$

As in the previous case, representing $a(t)$, $X(t)$ and $\tilde{a}(t)$, $\widetilde{X}(t)$ as algebraic polynomials and assuming $n1 = N1$ and $n2 = N2$ we get representations for errors $\Delta a(t)$, $\Delta X(t)$

$$\Delta a(t)=\sum_{i=1}^{N1}(\Delta a)_i\cdot t^{i-1},\quad \Delta X(t)=\sum_{j=1}^{N2}(\Delta\beta)_j\cdot t^{j-1}$$

and a non-linear equation for estimation errors (d′), in which constant values $(\Delta a)_i$, $(\Delta\beta)_j$ are already sought unknown.

The solution of equation (d′) relative to unknown$(\Delta a)_i$, $(\Delta\beta)_j$, and therefore the determination of the errors of estimates $\Delta a(t)$, $\Delta X(t)$ is carried out in the same way as described in paragraph 3.3.2, which is related to the linearization and introduction of intermediate generalized unknown.

In this case, the areas of integration in (d′) should be selected in such a way as to preserve the heterogeneity of the respective SLAE. That is, when using the first method of compilation of SLAE, the integration intervals in the compilation of SLES for equation (d′) (for definition $\Delta a(t)$, $\Delta X(t)$) must be offset relative to the integration, which were used in the formulation of SLAE to determine the estimates themselves $\tilde{a}(t)$, $\widetilde{X}(t)$.

CONCLUSION

From the content of the monograph, it is clear why the statement of the physically single-channel principle of invariance is given precisely in relation to measuring systems: this invariance principle allows us to solve the complex problem of measurement in theory and in practice, which has attracted the particular attention of specialists for many decades - the problem of exclusion of the influence of parametric effects on the accuracy of dynamic measurements.

However, the single-channel principle of invariance can be applied in any other area where the reaction of linear non-stationary dynamic objects to some input effect is required to determine the input effect itself. Such research objects exist in a large number in many fields of science and technology - astronomy, physics, chemistry, economics, etc. Therefore, turning to the language of the generalized concepts of “object”, “input”, “output” and taking into account the content of Section 4.2, where the application of the single-channel invariance principle to the dynamics of a linear non-stationary measuring system of an arbitrary structure (unstructured) was considered, it can be argued that according to the reaction of a linear object with concentrated parameters to some input effect, an extended inverse problem can be solved - the determination of the input effect and the pulse transient function characterizing the dynamic properties of the object.

The main general methodological issues related to the application of single-channel invariance principle in the dynamics of linear objects were discussed in this monograph. However, it is clear that the application of this principle in any particular area may have some specifics.

In clause 2.4, the construction of an invariance algorithm for one rather narrow class of nonlinear measuring systems is considered. The logical continuation of the research carried out in this work is the study of the possibilities of implementation of the single-channel principle of invariance in relation to some general models of nonlinear measuring systems. In this direction, it seems promising to use the Volterra functional series, which can serve as models of a fairly wide class of nonlinear systems, and are a natural generalization of models of linear systems.

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